ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

COMBINATORIAL TECHNIQUES AND ABSTRACT WITT RINGS II

ROBERT FITZGERALD AND JOSEPH YUCAS

1. Introduction and terminology. We will follow mainly the notation and terminology set up in [**2**]. Thus (R, G_R, B_R, q_R) is an abstract Witt ring as defined in [**4**] and recall that $q_R : G_R \times G_R \to B_R$ is a symmetric bilinear mapping with G_R and B_R being groups of exponent 2. G_R has a distinguished element -1 satisfying $q(a, -a) = 1$, and q_R satisfies

$$\begin{aligned} &\text{For all } a,b,c,d \in G_R, q_R(a,b) = q_R(c,d) \text{ implies} \\ &\text{there exists } x \in G_R \text{ with } q_R(a,b) = q_R(a,x) \\ (L) \qquad &\qquad\qquad = q_R(c,x) = q_R(c,d). \end{aligned}$$

Denote by Q_R the image of q_R in B_R and when there is no confusion write $G = G_R$, $B = B_R$, $q = q_R$ and $Q = Q_R$. For $a \in Q_R$ set $Q(a) = \{q(a,x) \mid x \in G_R\}$. Y_R will denote the collection $\{Q(a) \mid a \in G_R \backslash \{1\}\}$ and $\{Q_i\}_{i=1}^n$ is the collection of distinct elements of Y_R. For a subgroup of Q of B_R, the subgroup $\{x \in G_R \mid Q(x) \subseteq Q\}$ of G_R will be denoted by $H(Q)$. We let $H_i = H(Q_i)$ and $h_i = |H_i|$. The value set of $\langle 1, x\rangle$ is $D\langle 1, x\rangle = \{y \in G_R \mid q(-x, y) = 1\}$, and, for any subgroup K of G_R, let $\dot{K}$ denote $K \backslash \{1\}$. Finally set $g = |G_R|$.

In §2 construct a quotient quaternionic mapping $\overline{q}$ and give some technical conditions under which $\overline{q}$ satisfies (L). This quotient technique together with the counting technique of [**2**], proves to be quite useful in §3 where we classify Witt rings having a simple Hasse diagram. Specifically, we generalize Cordes' classification [**1**] of Witt rings with ≤ 4 quaternion algebras by classifying all non-degenerate Witt rings with $|Y_R| \leq 4$ (See [**3**;Chapter 5, §10]) for a statement and proof of Cordes' classification using the notation and terminology used here.)

2. Quotients. Let R be an arbitrary abstract Witt ring with associated linked quaternionic mapping $q : G_R \times G_R \to B_R$. For an arbitrary subgroup Q of B_R set $H = H(Q)$ and define $\overline{q} : G_R/H \times G_R/H \to B_R/Q$ by $\overline{q}(\overline{a}, \overline{b}) = q(a,b)Q$, where $\overline{a} = aH$.