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ON THE PARITY OF THE CLASS NUMBER OF THE FIELD OF q-TH ROOTS OF UNITY

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ABSTRACT. It is shown that the parity of the class number of the field $\mathbf{Q}(\zeta_q)$ of the q-th roots of unity over the rationals is odd whenever q and $p = (q-1)/2$ are primes and 2 is inert in the real subfield of p-th roots of unity over the rationals. As a consequence, the genus coincides with the spinor genus of the ring of integers in $\mathbf{Q}(\zeta_q)$ viewed as a lattice over the ring of integers in the real subfield.

Throughout this article, ζ_n denotes a primitive n-th root of unity, $k_n = \mathbf{Q}(\zeta_n)$ is the cyclotomic field of n-th roots of unity over the rationals $\mathbf{Q}$; k_n^+ denotes the real subfield of k_n; $\mathcal{O}_n^+$ and $\mathcal{O}_n^+$ are the rings of algebraic integers in k_n and k_n^+; $\mathbf{C}_n, \mathbf{C}_n^+$ are the class groups of the two rings; and their orders h_n, h_n^+ are the class numbers of the two fields. It is known that h_n^+ divides h_n with quotient h_n^-, the relative class number of k_n over k_n^+. Moreover, h_n is odd if and only if h_n^- is odd [**8**, Satz 45]. We prove in this note the following

THEOREM. *If q and $p = (q-1)/2$ are prime integers and 2 is inert in the real subfield of the cyclotomic field of p-th roots of unity over the rationals then the class number of the cyclotomic field of q-th roots of unity is odd.*

The study of the parity of the class number of number fields is motivated by the research of several authors. H. Hasse credits E. Kummer with the initial investigations on the parity of h_q, q a prime, based on a series of Kummer's papers between 1847 and 1870 (See [**8**] for a list of Kummer's articles and Satz 45 for extensions and refinements of Kummer's results to imaginary cyclic extensions of $\mathbf{Q}$). Kummer's work can be viewed as an analogue of Gauss' work on the 2 primary component of the class groups of binary quadratic forms for the binary quadratic lattice $\mathcal{O}_q$ over $\mathcal{O}_q^+$ (see Proposition below). In

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