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FUNDAMENTALS OF ANALYSIS OVER SURREAL NUMBERS FIELDS

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ABSTRACT. The Tarski Principle informs us that, concerning first order statements, all real-closed fields are created equal. Thus the field $\mathbf{R}$ of all real-numbers and the field $\mathbf{R}_0$ of all real-algebraic numbers have the same first order theory; however, their higher order theories are quite different. For example, $\mathbf{R}$ is Dedekind-complete and is a vast transcendental extension of its prime field, whereas $\mathbf{R}_0$ is not Dedekind-complete and is an algebraic extension of its prime field. The surreal number fields $\xi\mathbf{No}$ are all real-closed. They have extraordinary higher order properties, which allow one to do analysis over them, as we will see below.

0. Introduction. The construction of the class, On, of all von Neumann ordinals is, in many ways, quite similar to some of the most instructive constructions of the surreal numbers. Let us recall von Neumann's definition. (For convenience, let us work within Kelley-Morse set theory. See, e.g., [**11**, Chapter 2] for details.)

A class A will be called *ε-transitive* if, for all sets x, y, for which $x \in y$ and $y \in A$, then $x \in A$. A is called an *ordinal* if it and each element in it is ε-transitive. Let On be the class of all ordinals. It is easy to see that the empty set is an ordinal, which is defined to be 0. Given $\alpha \in On$, let $\alpha + 1$ be defined to be the union of α and $\{\alpha\}$; then $\alpha + 1$ is in On. Clearly $0, 1, \ldots, n$ are finite ordinals. Let ω be defined to be the union of the set of all finite ordinals. ω is the smallest infinite ordinal. For all α, $\beta \in On$, one and only one of the following hold: $\alpha \in \beta$, $\beta = \alpha$, or $\beta \in \alpha$. (See, e.g., [**11**, pp. 68-75] for proofs and details.) One defines $\alpha < \beta$ in On if $\alpha \in \beta$, and one finds that, under this ordering, On is a well-ordered class. $\beta \in On$ is called a *limit ordinal* if there is no $\alpha \in On$ such that $\beta = \alpha + 1$. β is called a *non-limit ordinal* if it is not a limit ordinal. Thus, for example, ω is a limit ordinal, whereas 2 is a non-limit ordinal.

One interesting property of the (von Neumann) ordinals is that they are canonical objects in set theory. This follows from the fact that there

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