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APPROXIMATING π WITH RAMANUJAN'S MODULAR EQUATIONS

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ABSTRACT. Ramanujan, in Chapter 19 of his Second Notebook gives a number of remarkable quintic and septic modular equations for the eta and related multipliers. Two are of particular interest because they are given in explicit solvable form. This allows us to derive explicit quintic and septic algorithms for pi. The quintic algorithm is especially attractive and its genesis is discussed in some detail.

1. Introduction. There is a close and beautiful relationship between the transformation theory of elliptic integrals and the efficient high precision approximation of pi. The three most recent record calculations are due to Bailey [**1**] (29.3 million digits in 1986) and to Kanada (33.5 million digits in 1986 and 133.5 million digits in early 1987). Each of these employed the following quartically converging iteration as one of the two algorithms used in the calculation. (It is now customary to declare a record only after a computation has been corroborated by a second calculation using a different algorithm - in these cases related quadratically converging iterations.)

Algorithm 1. Let $\alpha_0 := 6 - 4\sqrt{2}$ and $y_0 := \sqrt{2} - 1$. Let
(a) $y_{n+1} := (1 - \sqrt{1 - y_n^4})/(1 + \sqrt{1 - y_n^4})$ and let
(b) $\alpha_{n+1} := (1 + y_{n+1})^4 \alpha_n - 2^{2n+3} y_{n+1}(1 + y_{n+1} + y_{n+1}^2)$.
Then

$$0 < \alpha_n - \pi^{-1} < 16 \cdot 4^n e^{-2 \cdot 4^n \pi}.$$

Thirteen iterations suffice for Bailey's and Kanada's calculations. The algorithm is derived and discussed in [**4**].

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