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AN EXAMPLE OF A CHEBYSHEV SET, THE COMPLEX CASE

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ABSTRACT. Recently, the authors were asked by Professor Klaus Donner whether the set of matrices, the determinant of which in its absolute value is less than or equal to one, is Chebyshevian within the space of $n \times n$ matrices endowed with the l^2-operator norm. The answer is yes. Although our proof is elementary we think that the result and its proof are of general interest. In addition, there is one further nice example of a Chebyshev set.

We denote by $\mathbf{C}^{n\times m}$ the vector space of complex $n \times m$ matrices over $\mathbf{C}, n, m \in \mathbf{N}$, with elements $A, B, \ldots$ For $A \in \mathbf{C}^{n\times p}$ and $B \in \mathbf{C}^{p\times m}, n, m, p \in \mathbf{N}, A^*$ denotes the adjoint of A in $\mathbf{C}^{p\times n}$ and AB the matrix product of A and B in $\mathbf{C}^{n\times m}$. Instead of $\mathbf{C}^{n\times 1}$ we write $\mathbf{C}^n$ the vector space of complex column vectors, we also write $z, u, \ldots$ to denote its elements.

By l_n^2 we denote $\mathbf{C}^n$ endowed with the Euclidean norm $|\cdot|_2$, and by $\mathcal{L}(l_n^2)$ the vector space of linear transformations of l_n^2 into itself. $\mathcal{L}(l_n^2)$ can be identified with $\mathbf{C}^{n\times n}$ endowed with the l^2-operator norm $||\cdot||_2$ where $A \in \mathbf{C}^{n\times n}$ acts on $z \in \mathbf{C}^n$ via the matrix product Az. Since it will not lead to any ambiguity, in the following we shall write $\mathbf{C}^n$ and $\mathbf{C}^{n\times n}$ instead of l_n^2 and $\mathcal{L}(l_n^2)$, respectively.

For an $A \in \mathbf{C}^{n\times n}$, we denote its singular value decomposition (SVD) by $U\Sigma V^*, U$ and V are unitary matrices in $\mathbf{C}^{n\times n}$ and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Σ is uniquely determined by A, its elements are the so-called singular values of A. If A is non-singular, the UV^* is also uniquely defined.

THEOREM.
$$\mathcal{S} = \{S \in \mathbf{C}^{n\times n} : |\det S| \leq 1\}$$
is a Chebyshev set in $\mathbf{C}^{n\times n}$.

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