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A NOTE ON POSITIVE QUADRATURE RULES

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ABSTRACT. A classical problem in constructive function theory is the characterization of positive quadrature rules by linear combinations of orthogonal polynomials the roots of which determine the nodes of the formula. A complete characterization has been derived by F. Peherstorfer in 1984. In this note a variant to his approach will be discussed. It is the one-dimensional restriction of a characterization of interpolatory cubature formulae which might be of some general interest.

1. Introduction. We denote $\mathbf{P}$ the ring of real polynomials in one variable and by $\mathbf{P}[a,b]$ the restriction of $\mathbf{P}$ to $[a,b] \subseteq \mathbf{R}$. The linear space spanned by $\{1, x, x^2, \ldots, x^m\}$ will be denoted by $\mathbf{P}_m$.

Let

$$I : \mathbf{P}[a,b] \to \mathbf{R} : f \to I(f),\ I(1) = 1,$$

be a strictly positive linear functional, i.e., I is linear and $f \geq 0$ implies $I(f) > 0$ for all $f \in \mathbf{P}[a,b], f \neq 0$. Thus I represents those functionals usually studied in numerical integration.

We denote by p_i the orthogonal polynomials of degree i with respect to I, normalized such that the highest coefficient is 1, i.e.,

$$p_i = id^i + \sum_{j=0}^{i-1} \alpha_j id^j$$

such that $I(fp_i) = 0$ for all $f \in \mathbf{P}_{i-1}$. These polynomials satisfy the recursion formula

$$(1) \quad p_0 = 1,\ p_1 = id - \Gamma_0,\ p_{i+1} = (id - \Gamma_i)p_i - \Lambda_i p_{i-1}, i = 1, 2, \ldots,$$

where

$$\Gamma_0 = I(id),\ \Lambda_0 = 1,\ \Gamma_i = \frac{I(idp_i^2)}{I(p_i^2)},\ \Lambda_i = \frac{I(p_i^2)}{I(p_{i-1}^2)},\ i = 1, 2, \ldots.$$