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ON A THEOREM OF CELLINA FOR SET VALUED FUNCTIONS

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1. Introduction. By a *multifunction* Γ from a metric space X to a metric space Y, we mean a function that assigns to each $x \in X$ a nonempty subset $\Gamma(x)$ of Y. It is natural to identify Γ with its *graph* in $X \times Y$, the set $\{(x,y) : x \in X \text{ and } y \in \Gamma(x)\}$. In so doing, we can consider a variety of approximation problems with respect to Hausdorff distance in $X \times Y$. Two fundamental questions have been these: (1) when can we approximate a multifunction from above by a decreasing sequence of "continuous" multifunctions? (2) when can we approximate a multifunction by continuous single valued functions? All of the positive results with respect to these questions assume that the multifunctions be convex valued, for they ultimately depend on paracompactness arguments. With respect to the first question, the fundamental result is Hukuhara's Theorem [**9**], recently extended and sharpened by De Blasi [**4**] and De Blasi and Myjak [**5**]. Here, we are interested in the second question, where the fundamental result is due to Cellina [**3**]. Specifically, we show that Cellina's Theorem admits a converse precisely when X is locally compact and Y is complete, and we extend his result to continuous starshaped valued multifunctions.

2. Background material. We first recall the notion of Hausdorff distance between nonempty subsets of a metric space. For this purpose, we denote the union of all open ε-balls whose centers run over a subset E of a metric space $\langle X, d\rangle$ by $S_\varepsilon[E]$ (abbreviating $S_\varepsilon[\{x\}]$ by $S_\varepsilon[x]$). If F_1 and F_2 are nonempty subsets of X and for some $\varepsilon > 0$ both $S_\varepsilon[F_1] \supset F_2$ and $S_\varepsilon[F_2] \supset F_1$, then the *Hausdorff distance* h_d between them is given by the formula

$$h_d(F_1, F_2) = \inf\{\varepsilon : S_\varepsilon[F_1] \supset F_2 \text{ and } S_\varepsilon[F_2] \supset F_1\}.$$

Otherwise, we write $h_d(F_1, F_2) = \infty$. If we restrict h_d to the nonempty closed subsets of X, then h_d defines an infinite valued metric. Basic

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