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CONES IN THE GROUP ALGEBRA RELATED TO SCHUR'S DETERMINANTAL INEQUALITY

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ABSTRACT. Let $c : S_n \to \mathbf{C}$ be a complex valued function on the symmetric group. For $A = (a_{ij})$, an n-by-n matrix, define

$$d_c(A) = \sum_{\sigma \in S_n} c(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}.$$

Suppose C is the cone of all functions c such that $d_c(A) \geq 0$ for all positive semidefinite A (written $A \geq 0$). We show that $d_c(A) \geq c(e)\det(A)$ for all $c \in C$ and all $A \geq 0$, and then investigate the structure of C.

1. Introduction. Denote by H_n the cone of positive semidefinite hermitian n-by-n matrices. In 1893, J. Hadamard proved that $h(A) \geq \det(A)$ for all $A \in H_n$, where $h(A)$ is the product of the main diagonal entries of A. In 1918, I. Schur published a dramatic improvement of the Hadamard Determinant Theorem: Let G be a subgroup of the symmetric permutation group S_n. Suppose χ is an irreducible, complex character of G. If $A = (a_{ij})$ is an n-by-n matrix,define

$$d_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}. \tag{1}$$

In the recent literature, it has been customary to state Schur's Inequality as

$$d_\chi(A) \geq \chi(e)\det(A), \tag{2}$$

$A \in H_n$. As pointed out in [**1**], this inequality does not do justice to the full power of Schur's result. We will have more to say about this presently.

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