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MAPPINGS INTO SETS OF MEASURE ZERO

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ABSTRACT. Let f and g be functions of bounded variation on $[0, 1]$ and let λ denote Lebesgue outer measure. We give a necessary and sufficient condition that $\lambda gS = 0$ implies $\lambda fS = 0$, for all subsets $S \subset [0, 1]$. This condition is $\lambda fX = 0$, where X is a particular set depending on f and g.

In this paper, f and g are real valued functions of bounded variation on $[0, 1]$ and λ denotes Lebesgue outer measure. F and G are their total variation functions, $F(x) = V_0^x(f)$ and $G(x) = V_0^x(g)$ for $0 \leqq x \leqq 1$. We will give a necessary and sufficient condition that $\lambda gS = 0$ implies $\lambda fS = 0$, for any set $S \subset [0, 1]$. This condition is disclosed by the status of just one set determined by f and g. Our work will generalize and unify a number of more or less known corollaries concerning functions satisfying property N, absolutely continuous functions, saltus functions, and finite Borel measures on $[0, 1]$.

Define the set

$$X = \{x \in (0, 1)\colon \text{either } \lim_{h\to\infty} |(f(x+h) - f(x))/(g(x+h) - g(x))| = \infty \text{ or } x \text{ lies in the interior of the set } g^{-1}g(x)\}.$$

(Here we omit those h for which $g(x + h) = g(x)$.) We offer

THEOREM 1. *A necessary and sufficient condition that*

$$(*) \qquad \lambda fX > 0$$

holds is that there exists some set $S \subset [0, 1]$ *such that* $\lambda gS = 0 < \lambda fS$. *Moreover,* $\lambda gX = 0$ *whether* $(*)$ *holds or not.*

In other words, the question whether $\lambda gS = 0$ implies $\lambda fS = 0$, for all sets $S \subset [0, 1]$, is settled by the status of the one set, fX. Before developing a proof of Theorem 1, let us discuss some of its consequences. A

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