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CONNECTEDNESS PROPERTIES OF SUPPORT POINTS OF CONVEX SETS

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ABSTRACT. It is shown that the set of support points of certain convex subsets of a Banach space is $\mathscr{C}^\infty$.

Let E be a real Banach space and E^* its continuous dual. The natural pairing between these spaces will be denoted by $\langle x, x^* \rangle$ for $x \in E$ and $x^* \in E^*$. If $C \subseteq E$, we will write $M(x^*, C)$ in place of $\sup \{\langle x, x^* \rangle : x \in C\}$. The set of support points of C (written: supp C) is the collection of points $x \in C$ for which there exists $x^* \in E^* \backslash \{0\}$ such that

$$\langle x, x^* \rangle = M(x^*, C).$$

The set C is boundedly (weakly) compact if $C \cap B$ is (weakly) compact for each closed ball in E.

A space Y is said to be k-connected, if it is homotopically trivial over the k-dimensional sphere S^k. If Y is k-connected, for each $k = 0, \ldots, n$, then Y is said to be $\mathscr{C}^n$. An example, the n-dimensional Euclidean sphere S^n is $\mathscr{C}^{n-1}$ but not $\mathscr{C}^n$. A space is said to be $\mathscr{C}^\infty$ if it is $\mathscr{C}^n$ for every n.

If C is a closed convex subset of E, then supp C is known to be a norm dense F_σ subset of the boundary of C (written as bdry C). It is also known [4] that if C contains no hyperplane and is boundedly weakly compact, then supp C is connected.

We show here, that under these same assumptions, supp C is actually arcwise connected. In addition, we show that if C contains no linear variety of finite codimension, then supp C is $\mathscr{C}^\infty$. We also show that if C is boundedly compact, then supp C is contractible.

If $a \in C$, we will use the notation C_a for the union of all open half-spaces not containing C and which are determined by support functionals at a; that is

$$C_a = \bigcup \{(x^* > \langle a, x^* \rangle): x^* \neq 0, \langle a, x^* \rangle = M(x^*, C)\},$$

where $(x^* > \langle a, x^* \rangle) = \{x \in E: \langle x, x^* \rangle > \langle a, x^* \rangle\}$.

We also use the notation X_a for the set $a + \bigcup \{n(C - a): n \in N\}$, and (x, y) for the open line segment $\{\lambda x + (1 - \lambda)y: 0 < \lambda < 1\}$.

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