ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 4, Fall 1985

ON A THEOREM OF BERNSTEIN

M. A. MALIK

1. Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n and $P'(z)$ denote its derivative. Concerning the estimate of $|P'(z)|$ the following result is well known:

THEOREM A. *If $P(z)$ is a polynomial of degree n and* $\max_{|z|=1} |P(z)| = 1$ *then for $|z| \leqq 1$*

$$|P'(z)| \leqq n. \tag{1}$$

There is equality in (1) *if and only if $P(z) \equiv \alpha z^n$, $|\alpha| = 1$.*

Theorem A is known as Bernstein's Theorem. It can be deduced from a result (also known as Bernstein's Theorem) on the derivative of a trigonometric polynomial which can be proved following an interpolation formula obtained by M. Riesz [3]; from where it is also verified that equality in (1) holds only if $P(z) \equiv \alpha z^n$, $|\alpha| = 1$. In **[1]**, S. Bernstein proved the following generalization of Theorem A by the use of Gauss-Lucas Theorem; see also N. G. De Bruijn **[2]**:

THEOREM B. *Let $P(z)$ and $Q(z)$ be polynomials satisfying the conditions that $Q(z)$ has all its zeros in $|z| \leqq 1$ and the degree of $P(z)$ does not exceed that of $Q(z)$. If*

$$|P(z)| \leqq |Q(z)| \quad \text{on} \quad |z| = 1 \tag{2}$$

then

$$|P'(z)| \leqq |Q'(z)| \quad \text{on} \quad |z| = 1. \tag{3}$$

2. In this paper, we study the case when there is equality in (3). In fact, we prove:

THEOREM 1. *Let the hypothesis of* Theorem B *be satisfied. If there is equality in* (3) *at any point μ on $|z| = 1$ where $Q(\mu) \neq 0$ then $P(z) \equiv \alpha Q(z)$, $|\alpha| = 1$.*

Research supported by NSERC, Canada.
1980 subject classification 30A10; 30C10.
Keywords: Polynomials, derivative, inequality, etc.
Received by the editors on March 21 1984.