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ON THE GREATEST PRIME FACTOR OF TERMS OF A LINEAR RECURRENCE SEQUENCE

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In memory of R.A. Smith and E.G. Straus

The purpose of this note is to survey the results which have been obtained concerning the greatest prime factor and also the greatest square-free factor of terms of linear recurrence sequences. We shall first discuss the results which apply to general linear recurrence sequences. Next we shall consider binary recurrence sequences and finally, Lucas and Lehmer sequences.

For any integer n, let $P(n)$ denote the greatest prime factor of n with the convention that $P(0) = P(\pm 1) = 1$. Further, let $Q(n)$ denote the greatest square-free factor of n with the convention that $Q(0) = Q(\pm 1) = 1$. Thus, if $n = p_1^{h_1} \cdots p_r^{h_r}$ with $p_1, \ldots, p_r$ distinct primes and $h_1, \ldots, h_r$ positive integers, then $Q(n) = p_1 \cdots p_r$.

Let $r_1, \ldots, r_k$ and $u_0, \ldots, u_{k-1}$ be integers and put $u_n = r_1 u_{n-1} + \cdots + r_k u_{n-k}$, for $n = k, k+1, \ldots$. The sequence $(u_n)_{n=0}^{\infty}$ is a linear recurrence sequence. Denote the field of rational numbers by $\mathbf{Q}$. It is well known (see page 63 of [**14**]) that, for $n \geqq 0$,

$$u_n = f_1(n)\,\alpha_1^n + \cdots + f_t(n)\alpha_t^n, \tag{1}$$

where $f_1, \ldots, f_t$ are non-zero polynomials in n with degrees less than $\ell_1, \ldots, \ell_t$, respectively, and with coefficients from $\mathbf{Q}(\alpha_1, \ldots, \alpha_t)$, where $\alpha_1, \ldots, \alpha_t$ are the non-zero roots of the characteristic polynomial of the sequence, $x^k - r_1 x^{k-1} \cdots - r_k$, and $\ell_1, \ldots, \ell_t$ are their respective multiplicities. We shall say that the sequence $(u_n)_{n=0}^{\infty}$ is non-degenerate if $t > 1$, $f_i \neq 0$, for $1 \leqq i \leqq t$, and α_i/α_j, for $1 \leqq i < j \leqq t$, are not roots of unity.

In 1921, Polya [**22**] showed that if u_n is the n-th term of a non-degenerate linear recurrence sequence, then $\lim\sup_{n\to\infty} P(u_n) = \infty$. In 1935, Mahler

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