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THE FUNDAMENTAL LEMMA OF BRUN'S SIEVE IN A NEW SETTING

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This paper has emerged from the need experienced in [5] to estimate from above the number of lattice points in a four dimensional cube that remain after a sieving process. A new variant of Brun's upper sieve has been devised for this purpose, however, as was pointed out to the author by Dr M. Ram Murty the same upper bound could be obtained by the large sieve and this approach has been finally adopted in [5]. Pursuing further the small sieve approach one obtains a new variant of the fundamental lemma of Halberstam and Richert embodied in the following theorem.

THEOREM 1. *Let* $\mathscr{A}$ *be a finite set and* $\{\mathscr{T}_p\}$ *a family of sets indexed by primes from a certain set* $\mathscr{P}$. *Assume that for a certain multiplicative function* $\omega(d)$ *defined on all squarefree positive integers* d *and suitable real numbers* $X > 0$, $A_1 \geqq 1$, $A_2 \geqq 1$, A_3, $k \geqq 1$, κ *we have*

$$0 \leqq \frac{\omega(p)}{p} \leqq 1 - \frac{1}{A_1} \text{ for all primes } p, \tag{1}$$

$$\sum_{w \leqq p < z} \frac{\omega(p) \log p}{p} \leqq \kappa \log \frac{z}{w} + A_2 \text{ for all } w, z \text{ with } 2 \leqq w < z, \tag{2}$$

$$\Big|\Big|\mathscr{A} \cap \bigcap_{\substack{p \in \mathscr{P} \\ p \mid d}} \mathscr{T}_p\Big| - \frac{\omega(d)}{d} X\Big| \leqq A_3 X^{1-(1/k)} d^{k-1} \omega(d). \tag{3}$$

Then for all $z \leqq X$ *the number*

$$S(\mathscr{A}; \mathscr{P}, z) = \Big|\mathscr{A} \setminus \bigcup_{\substack{p \in \mathscr{P} \\ p < z}} \mathscr{T}_p\Big|$$

satisfies the relation

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