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CYCLOTOMY OF ORDER TWICE A PRIME

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Dedicated to the memory of E. G. Straus

Gauss defined f-nomial periods for a prime $p = ef + 1$ as

$$\eta_j = \sum_{r_i \in C_j} \zeta_p^{r_i} \text{ where } \zeta_p = \exp(2\pi_\delta i/p) \tag{1}$$

and C_j is the residue class with index j with respect to some primitive root g. These periods satisfy an irreducible monic period equation of degree e with integer coefficients

$$f_e(x) = \prod_{j=0}^{e-1} (x - \eta_j) = 0. \tag{2}$$

Kummer proved that if p is replaced by a general n then all the prime factors of the integers represented by $f_e(N)$, where N is any integer, are e-th power residues of p, except possibly when they divide P_k with $(e, k) = r \neq 1$, where

$$P_k = \prod_{i=0}^{e-1} (\eta_i - \eta_{i+k}) \tag{3}$$

in which case they may be only r-th power residues of p. Kummer [**3**] called such primes exceptional.

Recently Evans [**2**, p.13] proved Kummer's theorem for a generalized cyclotomy in which

$$\eta_j = \sum_{r \in C_j} \alpha_i \zeta_n^r \text{ with } \alpha_i \in \mathbf{Z}(\zeta_s),\ (s, n) = 1. \tag{4}$$

He also defined semiexceptional divisors as those divisors of the discriminant $D_e = \prod_{k=1}^{e-1} P_k$ that are not e-th powers residues and found for $e = 8$ some semiexceptional divisors which are not exceptional [**2**, p.22–24].

In a recent paper [**5**] we considered in great detail the special case of $e = 6$ and p a prime and found that all semiexceptional divisors are exceptional in this case. In doing this it became necessary to use a lemma derived from Theorem 5.2 of our paper [**4**] on Kloosterman sums

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