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ON THE DIVISOR SUM FUNCTION

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In memory of Robert A. Smith and Ernst G. Straus

1. Introduction. The function $\tau_k(n)$ representing the number of ways of expressing n as a product of k factors (the order of the factors being taken into account) has been studied since the time of Dirichlet. In contrast to this well-established function, the corresponding sum function $\sigma(n, k)$, which we define as the sum of the divisors corresponding to such factorizations of n, does not seem to have appeared in the literature. Indeed the only reference the authors can submit is their preliminary report [**9**].

We here formally define the divisor sum function $\sigma_r(n, k)$ for the rth powers of these divisors and obtain some identities (including two of a well-known Ramanujan type), and as an application obtain an asymptotic estimate for $\sum_{n\leqq x} \sigma_a(n, 3)\sigma_b(n, 3)$ which may be new. We extend the definition of $\sigma_r(n, k)$ to the case when k is complex and obtain some asymptotic estimates for its summatory function. Towards the end, we introduce the notation of k-ply perfect numbers and raise some open problems.

2. Preliminaries. Let

$$\tau_k(n) = \sum_{d_1 d_2 \cdots d_k = n} 1$$

for k a positive integer, so that $\tau_k(n)$ denotes the number of ways of expressing n as a product of k factors, the order of the factors being taken into account. In particular, let

$$\tau(n) = \tau_2(n) = \sum_{d_1 d_2 = n} 1 .$$

It is clear that if $\zeta(s)$ stands for the Riemann zeta function, we have

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s}, s = \sigma + it, \sigma > 1.$$

$\tau_k(n)$ is multiplicative in n, and if

$$n = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$$

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