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ON THE NORMAL NUMBER OF PRIME FACTORS OF $\varphi(n)$

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Dedicated to the memory of E. G. Straus and R. A. Smith

1. Introduction. Denote by $\Omega(n)$ the total number of prime factors of n, counting multiplicity, and by $\omega(n)$ the number of distinct prime factors of n. One of the first results of probabilistic number theory is the theorem of Hardy and Ramanujan that the normal value of $\omega(n)$ is loglog n. What this statement means is that for each $\varepsilon > 0$, the set of n for which

$$(1.1) \qquad |\omega(n) - \log\log n| < \varepsilon \log\log n$$

has asymptotic density 1. The normal value of $\Omega(n)$ is also loglog n.

A paricularly simple proof of these results was later given by Turán. He showed that

$$(1.2) \qquad \sum_{n \leqq x} (\omega(n) - \log\log x)^2 = x \log\log x + O(x)$$

from which (1.1) is an immediate corollary. The method of proof of the asymptotic formula (1.2) was later generalized independently by Turán and Kubilius to give an upper bound for the left hand side where $\omega(n)$ is replaced by an arbitrary additive function. The significance of the "loglog x" in (1.2) is that it is about $\sum_{p \leqq x} \omega(p)p^{-1}$, where p runs over primes. Similarly the expected value of an arbitrary additive function $g(n)$ should be about $\sum_{p \leqq x} g(p)p^{-1}$.

The finer distribution of $\Omega(n)$ and $\omega(n)$ was studied by many people, culminating in the celebrated Erdös-Kac theorem: for each $x \geqq 3$, u, let

$$G(x, u) = \frac{1}{x} \cdot \# \{n \leqq x : \Omega(n) \leqq \log\log x + u (\log\log x)^{1/2}\}.$$

Then

$$(1.3) \qquad \lim_{x \to \infty} G(x, u) = G(u) \overset{\text{def}}{=} (2\pi)^{-1/2} \int_{-\infty}^{u} e^{-t^2/2}\, dt,$$

the Gaussian normal distribution. The corresponding statement with

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