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APPENDIX

RICHARD ASKEY

This chapter is one of the most interesting in Ramanujan's Second Notebook and it shows Ramanujan's strengths and weaknesses better than any of the other chapters with the possible exception of the work on modular functions and forms that makes up much of the last part of this Notebook. First to the strengths, which can be illustrated both by reference to important work of Euler and Gauss, and also by some results that are unlikely to have been found by anyone else.

Gauss [12] defined two hypergeometric functions to be contiguous if they have the same power series variable, if two of the parameters are pairwise equal, and the third pair differ by one. Thus the functions contiguous to ${}_2F_1(a, b; c; x)$ are $F(a\pm) = {}_2F_1(a \pm 1, b; c; x)$, $F(b\pm)$ and $F(c\pm) = {}_2F_1(a, b; c \pm 1; x)$. Gauss showed that a hypergeometric function and any two contiguous to it are linearly related, and gave the fifteen formulas (actually nine different ones when the symmetry in a and b is used). These can be iterated, so any three hypergeometric functions whose parameters differ by integers are linearly related. Gauss used the linear relation between ${}_2F_1(a, b; c; x)$, ${}_2F_1(a, b + 1; c + 1; x)$ and ${}_2F_1(a + 1, b + 1; c + 2; x)$ to obtain the continued fraction in Entry 20.

Much earlier Euler [10] considered the integral

$$\int_0^1 (1 - xt)^{-a} t^{b-1}(1 - t)^{c-b-1}\, dt$$

and after integration by parts and a little algebra obtained a three term recurrence relation that he used to find a continued fraction. Later [11, vol. 2, §1, problem 130] he showed that

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(c - b)\,\Gamma(b)} \int_0^1 (1 - xt)^{-a} t^{b-1}(1 - t)^{c-b-1}\, dt.$$

Using this it is easy to see that his continued fraction is also an expansion of ${}_2F_1(a, b + 1; c + 1; x)/{}_2F_1(a, b; c; x)$. Surprisingly this continued fraction is not the same as Gauss's. What Euler had done was to derive the three term recurrence between ${}_2F_1(a, b; c; x)$, ${}_2F_1(a, b + 1; c + 1; x)$