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ON THE PIERCE-BIRKHOFF CONJECTURE

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Dedicated to the memory of Gus Efroymson

1. Introduction. In 1956, Birkhoff and Pierce [1] asked the question of characterizing the "ℓ-rings" and "f-rings" free on n generators, and conjectured that they should be rings of continuous functions on R^n, piecewise polynomials. The precise question known as the "Pierce-Birkhoff conjecture" is: given h: $\mathbf{R}^n \to \mathbf{R}$ continuous, piecewise polynomial, is h definable with polynomials by means of the operations sup and inf?

In a paper of Henriksen and Isbell [5] we can find explicit formulas showing that the set of such functions is closed under addition and multiplication, and so is a ring. We will call that ring ISD (Inf and Sup-definable).

Here we give a proof in the case $n = 2$ and make a study for the general case. G. Efroymson proved also this result independently and in a somewhat different way.

2. General Presentation. Given $P_1, \ldots, P_r \in \mathbf{R}[X_1, \ldots, X_n]$, let A_i be the semialgebraic subset of $\mathbf{R}^n$ defined by $h = P_i$. The point is to show that for any pair (i, j), there exists $e_{ij} \in$ ISD such that $e_{ij/A_j} \geqq P_{j/Aj}$ and $e_{ij/A_i} \leqq P_{i/A_i}$: if we get such functions, we have $h = \sup_j(\mathrm{Inf}_i(e_{ij}, P_j))$ and we are done.

So, let us complete the set $\{P_i - P_j\}_{i,j}$ in a separating family $\{Q_1, \ldots, Q_s\}$ [2] [4], which we can suppose made with irreducible polynomials.

All the functions considered being continuous, it is enough to work with the open sets of the partition which are the $\{x \in \mathbf{R}^n / \bigwedge\!\!\!\bigwedge_{i=1}^{s} Q_i \varepsilon_i\ 0\}$ with ε_i strict inequalities [such a set of disjoint open sets whose union is dense in R^n will be called "open partition" of $\mathbf{R}^n$]. Let us call again $(A_i)_{i=1}^{p}$ these open sets:

We get three possibilities for the pair (A_i, A_j):

1) $\bar{A}_i \cap \bar{A}_j = \phi$
2) $\mathrm{codim}(\bar{A}_i \cap \bar{A}_j) = 1$