ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

EXTENSIONS OF SEMIDEFINITE FUNCTIONS

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Dedicated to the memory of Gus Efroymson

Gondard and Ribenboim [4] proposed the following problem: let $V \subseteq \mathbf{R}^n$ be an algebraic set, $f \in \mathbf{R}[x]$, $\mathbf{x} = (x_1, \ldots, x_n)$, such that $f/_V \geqq 0$. Does there exist $F \in \mathbf{R}[x]$, $F \geqq 0$ over $\mathbf{R}^n$, such that $F/_V = f/_V$? They gave a partial positive answer. We prove a stronger result.

THEOREM 1. *Let $V \subset \mathbf{R}^n$ be an algebraic set, $f \in \mathbf{R}[x]$ with $f/_V \geqq 0$ and $\{a \in \mathbf{R}^n : f(a) = 0\} \cap V_c = \emptyset$ (V_c is the locus of central points of V). Then, there exists $F \in \mathbf{R}[x]$ non negative over $\mathbf{R}^n$ such that $F/_V = f/_V$. Moreover, if $f/_V > 0$, F is positive over $\mathbf{R}^n$.*

In the same paper [4] it is proved that the answer to the problem of extending f, if $f/_V \geqq 0$, is negative in general. For example, let us take $V = \{y^2 - x^3 = 0\}$ and $f = x$ However, we have been able to prove the following theorem.

THEOREM 2. *If $f \in \mathbf{R}[x]$ and $f/_V \geqq 0$, there exists an odd positive integer m and $F \in \mathbf{R}[x]$ non negative over $\mathbf{R}^n$, verifying $F/_V = f^m/_V$*

We look at these problems in two different ways.

A) We restrict ourselves to the case where $V \subset \mathbf{R}^n$ is a curve.

B) Given $f \in \mathbf{R}[x]$ such that $f/_V \geqq 0$, does there exist

i) A polynomial $F \in \mathbf{R}[x]$

ii) A regular function F

iii) A rational function F

such that $F \geqq 0$ (where F is defined) and $f/_V = F/_V$?

If we denote by $D(f)$ the "bad set" of f (see [3]), we prove the following.

PROPOSITION 3. *Condition* ii) *is equivalent to condition* iii), *and both are implied by $D(f) = \emptyset$.*

So, we are concerned with the existence of polynomial extensions (condition i)) or regular extensions (condition ii))

Since $\operatorname{codim}_V D(F) \geqq 2$ when $V \subset \mathbf{R}^n$ is a normal algebraic set, ([3]), we conclude the following proposition.

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