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BASIC HOMOTOPY THEORY OF LOCALLY SEMIALGEBRAIC SPACES

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Dedicated to the memory of Gus Efroymson

We use the notations of our survey article [1] on locally semialgebraic spaces in this volume. We want to indicate that it is possible to develop a homotopy theory for locally semialgebraic spaces over an arbitrary real closed field R which works as efficiently as the homotopy theory for topological spaces. For this purpose we give the basic definitions and some results.

NOTATION. Let M, N be locally semialgebraic spaces over R and $A_1, \ldots, A_r$ resp. $B_1, \ldots, B_r$ be locally semialgebraic subsets of M resp. N. By $[(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)]$ we denote the set of homotopy classes of locally semialgebraic maps $(M, A_1, \ldots, A_r) \to (N, B_1, \ldots, B_r)$ (cf. [1 Definition 1 in §4]). In the case $R = \mathbf{R}$ we denote by $[(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)]_{top}$ the set of continuous homotopy classes of continuous maps between the associated systems of topological spaces $(M_{top}, A_{1,top}, \ldots, A_{r,top}), (N_{top}, B_{1,top}, \ldots, B_{r,top})$.

MAIN THEOREM 1. *Let $\tilde{R}$ be a real closed field extension of R. Let M be an affine semialgebraic space and N be a regular locally semialgebraic space over R. For closed semialgebraic subsets $A_1, \ldots, A_r$ of M and locally semialgebraic subsets $B_1, \ldots, B_r$ of N the natural map* $[(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)] \to [(M(\tilde{R}), A_1(\tilde{R}), \ldots, A_r(\tilde{R})), (N(\tilde{R}), B_1(\tilde{R}), \ldots, B_r(\tilde{R}))]$ *which maps the class $[f]$ of a map f to the class $[f_{\tilde{R}}]$ of the base extension $f_{\tilde{R}}$ of f* (cf. [1] Example 1.7) *is bijective.*

Unfortunately we do not yet know whether Theorem 1 is also true if M is not affine semialgebraic. Our proof uses, besides the method of simplicial approximation, semialgebraic mapping spaces and Tarski's principle, and these techniques are restricted to the affine semialgebraic setting. The situation is slightly better when we compare semialgebraic homotopy sets with topological homotopy sets in the case $R = \mathbf{R}$.

MAIN THEOREM 2. *Let M, N be regular locally semialgebraic spaces over*