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A NOTE ON ORDER CONVERGENCE IN COMPLETE LATTICES

H. DOBBERTIN, M. ERNÉ AND D. C. KENT

ABSTRACT. Order convergence is studied in a complete lattice L_1 which is the image of another complete lattice L under a complete homomorphism. The goal is to relate order convergence in L to that in L_1. For instance, we show that order convergence in L_1 is pretopological if it is pretopological in L, while topological order convergence is in general not preserved under complete images. We conclude with some applications and examples.

1. Complete homomorphisms and order convergence. Throughout this note, let L and L_1 be complete lattices and suppose that $\varphi: L \to L_1$ is a surjective, complete homomorphism (i.e., $\varphi(L) = L_1$ and φ preserves arbitrary sups and infs). The lower adjoint $\varphi_*: L_1 \to L$ and the upper adjoint $\varphi^*: L_1 \to L$ of φ are given by

$$\varphi_*(y) = \inf\{x \in L: \varphi(x) = y\},$$
$$\varphi^*(y) = \sup\{x \in L: \varphi(x) = y\}$$

$(y \in L_1)$.

The following facts are well-known (see, e.g., [3] or [6] for a general theory of adjoint pairs):

$$\text{(i) } \varphi_*\varphi(x) \leqq x \text{ and } \varphi^*\varphi(x) \geqq x, \qquad x \in L,$$
$$\text{(ii) } \varphi\varphi_*(y) = y \text{ and } \varphi\varphi^*(y) = y, \qquad y \in L_1,$$
$$\text{(iii) } \varphi_* \text{ preserves sups, and } \varphi^* \text{ preserves infs.}$$

We set $[a) = \{x \in L: a \leqq x\}$, $(b] = \{x \in L: x \leqq b\}$, and $[a, b] = [a) \cap (b]$, for all $a, b \in L$. Sets of this form are called intervals. For $X \subseteq L$, $\downarrow X = \bigcup\{(x]: x \in L\}$ denotes the lower set generated by X.

A (set-theoretical) filter $\mathfrak{F}$ on L order converges to a point $x \in L$, written $\mathfrak{F} \to_L x$, if

$$x = \inf\{\sup F: F \in \mathfrak{F}\} = \sup\{\inf F: F \in \mathfrak{F}\}.$$

By the order convergence $\mathbf{O}(L)$ of L we mean the set of all pairs $(\mathfrak{F}, x)$

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