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AN APPLICATION OF MULTIVARIATE B-SPLINES TO COMPUTER-AIDED GEOMETRIC DESIGN

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ABSTRACT. A multidimensional analogue of Schoenberg's Spline Approximation Method is presented within a Computer-Aided Design context. The construction uses multivariate *B*-splines to approximate real valued functions on arbitrary polyhedrons in Euclidean spaces of any dimension.

1. Introduction. Univariate *B*-spline approximation techniques have enjoyed an immense popularity as a tool in the field of Computer-Aided Geometric Design (CAGD). Most applications utilize the Spline Approximation Method of Schoenberg and Greville [15] in which given a sequence of knot points $t_0 < \cdots < t_{N+k+1}$ and a real valued function f defined on $[t_0, t_{N+k+1}]$ the approximation to f is defined by

$$(1.1) \qquad \mathscr{M}_k(f: x) = \sum_{i=0}^{N} f(u_i)\, N(x: t_i, \ldots, t_{i+k+1}).$$

In this expression the $u_i = (t_{i+1} + \cdots + t_{i+k})/k$ are called the nodes and the $N(x: t_i, \ldots, t_{i+k+1})$ are the normalized *B*-spline basis functions defined on the knots $t_i, \ldots, t_{i+k+1}$ of degree k. In CAGD the coefficients $f(u_i)$ are frequently supplied a priori and are the "graphical handles" used to interact with a design. Within this context the function f is chosen to be a piece-wise linear function parametrized so that the coefficients are the breakpoints at the nodes (see Figure 1.1).

This approximation method possesses a number of properties which make it particularly attractive for use in CAGD. First, the method is a local one in that for any evaluation there are at most m nonzero basis functions meaning that small perturbations to the function f result in localized changes to $\mathscr{M}_k(f)$. A second important feature of this scheme is that the basis functions form a partition of unity which has the desirable consequences that the approximation lies within the convex hull of the coefficients of each basis function as well as invariance under rigid motion transformations. The last feature (1.1) possesses is that the approximation oscillates no more than the function f itself. Note, in particular, that this constrains $\mathscr{M}_k(f)$ to reproduce linear functions and in fact the choice of nodes in (1.1) was made so that it would do just that.