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ON THE PICARD GROUP OF A COMPACT COMPLEX NILMANIFOLD

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1. Introduction. This paper deals with compact complex nilmanifolds. By a nilmanifold we mean a homogeneous space of a nilpotent Lie group. The nilmanifold we consider arises as the compact quotient of a simply connected nilpotent Lie group G by a lattice Γ of G. We write G/Γ to denote such a space. In general, G/Γ is a non-Kähler manifold, and in fact, it is Kähler if and only if it is a complex torus (see [5]). However, G/Γ is a generalization of the torus, and to this end, there is a canonically associated torus T given by

$$T = G/[G, G]/\pi(\Gamma), \tag{1.1}$$

where $G/[G, G]$ is a vector group and $\pi(\Gamma)$ is a lattice of $G/[G, G]$, π: $G \to G/[G, G]$ being the projection map. T plays an important role in the analysis of G/Γ. We point out that there is a holomorphic fibration of G/Γ over T where the fibre is the compact complex nilmanifold $N_1 = [G, G]/\Gamma_1$, $\Gamma_1 = \Gamma \cap [G, G]$. We let π: $G/\Gamma \to T$ also denote the bundle map.

Our main purpose is to give a description of the Picard group of G/Γ; that is, $\mathrm{Pic}(G/\Gamma)$, the group of holomorphic isomorphism classes of holomorphic line bundles on G/Γ. To this end, we obtain a partial generalization of the Appell-Humbert Theorem from the case of the complex torus to the case of G/Γ. Sakane [4] has shown that the first Chern class of any holomorphic line bundle $\mathscr{L}$ on G/Γ, $c_1(\mathscr{L})$, is represented by a unique hermitian form H defined on $G/[G, G]$. As a consequence of the Appell-Humbert Theorem, we know that H corresponds to the first Chern class of a line bundle on the complex torus T if and only if the imaginary part of H, A, is integral on the lattice $\pi(\Gamma)$. Consequently, we can factor $\mathscr{L}$ as

$$\mathscr{L} = \mathscr{L}_\lambda \otimes \pi^*\mathscr{L}_1, \tag{1.2}$$

where $\mathscr{L}_\lambda$ is the line bundle associated to some character λ of the lattice Γ and $\pi^*\mathscr{L}_1$ is the pullback of a line bundle $\mathscr{L}_1$ on T with $c_1(\mathscr{L}_1)$ determined by H. See Theorem 3 for details.

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