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ARENS REGULARITY OF CONJUGATE BANACH ALGEBRAS WITH DENSE SOCLE

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ABSTRACT. Let A be a semi-simple Banach algebra which is isometrically isomorphic to the conjugate space of a Banach space V. Suppose that A is weakly completely continuous (w.c.c.). We first show that Arens regularity of A can be obtained by imposing certain conditions on V. If, moreover, A has dense socle, we show that these conditions on V can be obtained in turn by demanding that the maximal modular left (right) ideals and minimal idempotents of A have certain properties.

Introduction. Let A be a Banach algebra which is isometrically isomorphic to the conjugate space of a Banach space V. Identify V as a subspace of A^*. By a theorem of Dixmier [4], $A^{**} = \pi(A) \oplus V^\perp$. (See notation in §2.) Hence for A to be Arens regular the two Arens products must agree on $V^\perp$. For w.c.c. Banach algebras this is connected with the A-invariance of V. We obtain conditions for V to be A-invariant. Our conditions involve the concept of an HB-subspace used by Hennefeld [8]. We apply this concept also to the space V and come up with the notion of a VHB-subspace. The subspaces of A which we want to be VHB-subspaces are the maximal modular left (right) ideals. Examples of Banach algebras which have maximal modular left (right) ideals that are HB-subspaces or VHB-subspaces are discussed in §3. In §4 we show a connection between A-invariance of V and VHB-subspaces and present several results on Arens regularity of conjugate Banach algebras.

2. Preliminaries. Let A be a Banach algebra and let A^* and A^{**} be its first and second conjugate spaces. The two Arens products on A are defined in stages as follows [1]. Let $x, y \in A$, $f \in A^*$ and $F, G \in A^{**}$. Define $f \circ x \in A^*$ by $(f \circ x)(y) = f(xy)$, $F \circ f \in A^*$ by $(F \circ f)(x) = F(f \circ x)$, $F \circ G \in A^{**}$ by $(F \circ G)(f) = F(G \circ f)$, $x \circ' f \in A^*$ by $(x \circ' f)(y) = f(yx)$, $f \circ' F \in A^*$ by $(f \circ' F)(x) = F(x \circ' f)$, and $F \circ' G \in A^{**}$ by $(F \circ' G)(f) = G(f \circ' F)$. Then A^{**} is a Banach algebra under the Arens products $F \circ G$ and $F \circ' G$. Both of these products extend the original multiplication on

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