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A COINCIDENCE THEOREM IN CONVEX SETS WITH APPLICATIONS TO PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. Variations of the expansion and compression theorems of Krasnosel'skii have been used frequently (see, for example, [1], [3], [4], [5], [6], [8], [9]) to obtain existence of solutions to various problems involving ordinary and functional differential equations where the solution is required to lie in some cone. These theorems apply to operator equations of the form $x = Ax$.

In §2 of this paper we establish a more general framework to treat equations of the form $Lx = Nx$ where L is not necessarily invertible; we obtain Theorem 2.3 as a very simple consequence of invariance under homotopy of the Leray-Schauder degree. This theorem is in the same spirit as the continuation theorem of Mawhin [7] for coincidence degree.

In §3 we illustrate the use of this theorem by giving conditions (Theorems 3.1 and 3.2) under which the problem $\dot{x}(t) = f(t, x(t))$, $x(0) = x(1)$, has a nonzero solution $x(t)$ satisfying $x(t) \geqq 0$.

Finally, in §4 we discuss the problem $\ddot{x}(t) = f(t, x(t))$, $x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0$ where we again seek non-negative solutions.

2. A coincidence theorem for convex sets. Let X and Z be real Banach spaces. We will consider a linear mapping $L\colon \operatorname{dom} L \subset X \to Z$ and a not necessarily linear mapping $N\colon X \to Z$ with the following properties.

a) L is Fredholm of index 0. This entails that $\operatorname{Im} L$ be closed and that $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L$. As a consequence of this property there exist continuous projection mappings $P\colon X \to X$ and $Q\colon Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. These projections induce a decomposition of X and Z into corresponding subspaces as indicated in the following diagram.

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