ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 3, Summer 1982

COMPUTABLE ERROR BOUNDS FOR FINITE ELEMENT APPROXIMATIONS TO THE DIRICHLET PROBLEM

ROBERT E. BARNHILL AND CALVIN H. WILCOX

ABSTRACT. The constants bounding the solution of Poisson's equation in terms of the given boundary data are derived. Knowledge of these constants then permits the interpolation remainder theory of Barnhill and Gregory to be used to find computable finite element error bounds.

1. Introduction. The purpose of this paper is to extend the finite element error bounds of Barnhill and Gregory [**2**, **3**] so that they become numerically computable from the data. This section is a very brief review of the necessary facts.

DIRICHLET PROBLEM. If $\Omega \subset \mathbf{R}^2$ is a bounded domain, find $u: \Omega \to \mathbf{R}$ such that

$$\begin{aligned} \Delta u &= f \text{ in } \Omega \ (f \text{ given}) \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

WEAK FORMULATION. Find $u \in \mathring{W}_2^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega fv dx \qquad \forall v \in \mathring{W}_2^1(\Omega).$$

OPERATOR-THEORETIC FORMULATION. Define an unbounded linear operator

$$\Delta_D: L_2(\Omega) \to L_2(\Omega)$$

by

$$\begin{aligned} D(\Delta_D) &= \mathring{W}_2^1(\Omega) \cap \{u: \Delta u \in L_2(\Omega)\} \\ \Delta_D u &= \Delta u \ \forall u \in D(\Delta_D). \end{aligned}$$

LEMMA 1. *u is a weak solution of the Dirichlet problem with $f \in L_2(\Omega)$ if and only if $u \in D(\Delta_D)$ and $\Delta_D u = f$.*

PROOF. ($\Rightarrow$) Take $v \in C_0^\infty(\Omega)$. ($\Leftarrow$) $C_0^\infty(\Omega)$ is dense in $\mathring{W}_2^1(\Omega)$.

Received by the editors on January 29, 1981.