ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 3, Summer 1982

LIMITS OF DIRICHLET FINITE FUNCTIONS ALONG CURVES

MOSES GLASNER AND RICHARD KATZ

Let R be a noncompact connected orientable real analytic Riemannian n-manifold. In formulating the Dirichlet principle in the absence of a border for R various types of behavior at the ideal boundary have been used. Brelot [1] considered limits of functions along a family of curves tending to the ideal boundary, the collection of Green's lines. Royden [11] introduced a compactification R^* of R to which Dirichlet finite functions extend continuously and considered values of functions on Δ, the harmonic part of $R^*\backslash R$, as boundary values. Nakai [6], [7] showed that for Dirichlet finite functions these two modes of behavior are in a sense the same. Subsequently Ohtsuka [8] used limits along arbitrary curves tending to the ideal boundary and extremal length to specify boundary behavior. Since Δ is a tractable analytic device and extremal length is related to the geometry of R, it is important to determine the connection between the latter sort of boundary behavior and the former ones.

Let $\tilde{M}(R)$ denote the space of Tonelli functions on R with finite Dirichlet integrals, $D_R(f) = \int_R df \wedge * df < +\infty$. We shall show that an $f \in \tilde{M}(R)$ has limit 0 along almost every curve (in the sense of extremal length) joining a fixed parametric ball to the ideal boundary if and only if the values of f on Δ are 0. In particular, given a function $g \in \tilde{M}(R)$ the solution to the Dirichlet problem having the same boundary values as g does not depend on which of the above meanings is assigned to the term boundary values. As an application we give a criterion for R to carry nonconstant Dirichlet finite harmonic functions.

1. We begin by organizing some terminology for later use. We say that an open set $\mathcal{O} \subset R$ is an *end of R* if the relative boundary $\partial\mathcal{O}$ is piecewise smooth and compact in R whereas $\bar{\mathcal{O}}$ is not compact. A region $\Omega \subset R$ will be called *regular* if $\bar{\Omega}$ is compact and $\partial\Omega$ is piecewise smooth. The *relative harmonic measure* of an end $\mathcal{O}$ of R, $\omega = \omega(\cdot\,; \mathcal{O}, R)$ is defined as follows. Let $\{R_m | m = 1, 2, \ldots\}$ be an exhaustion of R by regular regions with $\partial\mathcal{O} \subset R_1$ and let $\omega_m = \omega_m(\cdot\,; \mathcal{O}, R)$ be in $\tilde{M}(R)$ such that $\omega_m | R\backslash\mathcal{O} = 0$, $\omega_m | \mathcal{O}\backslash R_m = 1$ and $\omega_m | \mathcal{O} \cap R_m$ is harmonic. By the maximum principle $0 \leqq \omega_{m+1} \leqq \omega_m \leqq 1$. Hence by the Harnack principle we may define

Received by the editors on January 26, 1981.
Copyright © 1982 Rocky Mountain Mathematics Consortium