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STRENGTHENED MAXIMAL FUNCTIONS AND POINTWISE CONVERGENCE IN R^n

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1. **Introduction.** Questions relating to the pointwise a.e. convergence of a sequence of operators applied to a function in L^p are usually handled in terms of some maximal function which serves as a pointwise majorant for all terms in the sequence. A simple example of this is Lebesgue's differentiation theorem in R^n, where we are concerned with a sequence of averages of a function f over balls centered at x. The Hardy-Littlewood maximal function arises naturally in this problem and provides the key to this theorem as well as many other problems of pointwise convergence; see Stein [3].

If instead of taking averages of f over balls at x we take averages over more general sets, then a number of very interesting problems arise. See Guzmán [1] for a survey. Of course, for the averages to approach $f(x)$ we must require that the sets shrink to $\{x\}$ in some sense. For bounded continuous f little else in needed, but for f merely integrable the sets must shrink to $\{x\}$ regularly: the measure of each set in the sequence must be comparable to that of a ball centered at x and containing the set. For functions in L^p for $1 < p < \infty$ one expects some intermediate regularity condition to suffice; we develop such conditions here. We introduce set functions which measure the extent to which a set is concentrated near x; the appropriate regularity condition is to require that this set function be bounded by a multiple of Lebesgue measure on the sequence of sets considered. Our regularity condition is sufficiently general to allow us to deal with averages over unbounded sets.

In the process we introduce some new maximal functions which are useful for estimating convolution operators. We obtain estimates of the form

$$|K * f(x)| \leqq \|K\| F(x)$$

where F depends only on f. Such estimates are particularly useful when K depends on a parameter. In particular, when $K_\lambda(x) = \lambda^n K(\lambda x)$, the norms we introduce for K have the important property that $\|K_\lambda\| = \|K\|$. Consequently, we obtain some new sufficient conditions for pointwise

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