ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 1, Winter 1981

THE CLASS NUMBER OF $Q(\sqrt{-2p})$ MODULO 8, FOR $p \equiv 5$ (MOD 8) A PRIME

KENNETH S. WILLIAMS*

ABSTRACT. Let $p \equiv 5 \pmod 8$ be a prime. Let $h(\pm 2p)$ denote the class number of the quadratic field $Q(\sqrt{\pm 2p})$. Let $T + U\sqrt{2p}$ be the fundamental unit of $Q(\sqrt{2p})$. It is shown that $h(-2p) \equiv h(2p) + 2T + 2 \pmod 8$.

If p is a prime congruent to 5 modulo 8, it is well known that the class number $h(-2p)$ of the imaginary quadratic field $Q(\sqrt{-2p})$ is congruent to 2 modulo 4 (see for example [**2**: p. 413]). In this paper, we determine $h(-2p)$ modulo 8. This is a problem of D.H. Lehmer [**6**: p. 10]. (The corresponding problem for $h(-p)$, $p \equiv 3 \pmod 4$, has been solved by the author in [**9**].)

We let $\varepsilon_{2p} = T + U\sqrt{2p}$ be the fundamental unit of the real quadratic field $Q(\sqrt{2p})$, so that T and U are positive integers. It is a classical theorem of Dirichlet [**5**: p. 226] that ε_{2p} has norm -1, that is,

$$N(\varepsilon_{2p}) = T^2 - 2pU^2 = -1,$$

from which it follows that T and U are both odd. The class number $h(2p)$ of $Q(\sqrt{2p})$ is also congruent to 2 modulo 4 (see for example [**3**: p. 101]). With this notation we prove the following theorem.

THEOREM. $h(-2p) \equiv h(2p) + 2T + 2 \pmod 8$.

PROOF. It is assumed throughout that p is a prime congruent to 5 modulo 8. We set $\rho = \exp(2\pi i/p)$. For z a complex variable, we let

$$F_+(z) = \prod_{\substack{j=1 \\ (j/p)=+1}}^{p-1} (z - \rho^j), \quad F_-(z) = \prod_{\substack{j=1 \\ (j/p)=-1}}^{p-1} (z - \rho^j), \tag{1}$$

so that

$$F_+(z)F_-(z) = F(z), \tag{2}$$

where $F(z)$ is the cyclotomic polynomial of index p,

MOS Subject Classification Numbers: 12A25, 12A50.

Key words and phrases: Quadratic fields, class number, fundamental unit, Dirichlet class number formula.

*Research supported by National Research Council of Canada Grant No. A-7233.

Received by the editors on August 1, 1978.