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ON SOME PROBLEMS RELATED TO THE EXTENDED DOMAIN OF THE FOURIER TRANSFORM

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This note could have been written by N. Aronszajn (to whom the credit is due for the result in Theorem 1), and it is only proper that it should be dedicated to him.

Our objective here is to prove that the extended domain of the Fourier transform as defined in [1], [2] is the largest solid F space of functions whose Fourier transforms (in the sense of distributions) are functions. To make this statement precise, we have to recall some facts from [1] and [2].

If (X, dx) is a σ-finite measure space then $\mathfrak{M}(X)$ denotes the space of all measurable finite a.e. functions on X with the metric topology of convergence in measure on all subsets of X of finite measure. If (X, dx), (Y, dy) are two σ-finite measure spaces and $K \in \mathfrak{M}(X \times Y)$, then the *integral transformation* with the kernel K is a mapping from $\mathfrak{M}(X)$ into $\mathfrak{M}(Y)$ defined by $Kf(y) = \int_X K(x, y)f(x)\, dx$. The *proper domain* of K is defined by $\mathscr{D}_K = \{f \in \mathfrak{M}(X) : \int_X |K(x, y)f(x)|\, dx < \infty \text{ a.e.}\}$. K is *nonsingular* if there is $f \in \mathscr{D}_K$ such that $f > 0$ a.e.

The symbol $\subset_c$ will denote the continuous inclusion.

If A is an F-space, $A \subset_c \mathfrak{M}(X)$ then K is *A-semi-regular* (A-s.r.) if 1) $A \cap \mathscr{D}_K$ is dense in A, 2) the restriction $K|_{A \cap \mathscr{D}_K}$ is continuous from $A \cap \mathscr{D}_K$ (with the topology of A) into $\mathfrak{M}(Y)$. If K is A-s.r. then K can be extended to a continuous linear transformation $K_A : A \to \mathfrak{M}(Y)$ (which may no longer be an integral transformation).

An F-space $A \subset \mathfrak{M}(X)$ is *solid* provided $f \in A$, $g \in \mathfrak{M}(X)$, $|g(x)| \leqq |f(x)|$ a.e. imply $g \in A$. The class of solid F-subspaces of $\mathfrak{M}(X)$ we denote by FL.

The following result is quoted from [1].

PROPOSITION 1. *If K is a nonsingular integral transformation, then there exists an FL-subspace of $\mathfrak{M}$, denoted by $\tilde{\mathscr{D}}_K$, with the following properties:*

1) *K is $\tilde{\mathscr{D}}_K$-s.r.; denote $\tilde{K} = K_{\tilde{\mathscr{D}}_K}$,*
2) *If $A \in FL$ and K is A-s.r. then $A \subset_c \tilde{\mathscr{D}}_K$ and $K_A = \tilde{K}|_A$.*

$\tilde{\mathscr{D}}_K$ is referred to as *the extended domain* of K.

We turn now to the case of the Fourier transform; here $X = Y = \mathbf{R}^1$, $\mathfrak{M}(X) = \mathfrak{M}(\mathbf{R}^1) = \mathfrak{M}$, $K(x, y) \equiv \mathfrak{F}(x, y) = (2\pi^{-1/2} e^{-ixy}$, dx is the Lebesgue measure and the corresponding integral transform we denote by $\mathfrak{F}$. We

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