ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 4, Fall 1979

A NOTE ON MODULUS OF APPROXIMATE CONTINUITY ON $R(X)$

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1. Let X be a compact subset of the plane. We denote by $R(X)$ the uniform closure of $R_0(X)$, the set of rational functions having no poles on X. We say that ϕ is an *admissible function* if (a) ϕ is a positive, non-decreasing function defined on $(0, \infty)$ and (b) $\psi(r) = r/\phi(r)$ is also non-decreasing and $\lim_{r\to 0^+} r/\phi(r) = 0$.

Fix $x \in X$. Suppose ϕ is an admissible function and $\phi(0^+) = 0$. We say that the unit ball of $R(X)$ admits ϕ as a *modulus of approximate continuity at* x if

$$|f(y) - f(x)| \leqq \phi(|y - x|) \text{ for all } f \in R(X),\ \|f\| \leqq 1$$

and all y in a subset having full area density at x. Some properties concerning the modulus of approximate continuity have been investigated in [5] and [6]. It is known, for instance, at a non-peak point x, there exists an admissible function ϕ with $\phi(0^+) = 0$ such that the unit ball of $R(X)$ admits $\epsilon\phi$ as a modulus of approximate continuity at x, for every $\epsilon > 0$.

One can define a fractional order bounded point derivation in terms of representing measure, analytic capacity and modulus of approximate continuity respectively. However, it turns out that the definitions are not equivalent (see [6]).

Although the existence of modulus of approximate continuity at a point is in general a weaker condition than some other properties, we will show that it does imply that X has more than full area density at that point (Corollary 3).

Let E be a bounded plane set and denote by $H(E)$ the set of functions holomorphic off a compact subset of E, bounded in modulus by one, which vanish at ∞. The analytic capacity of E is $\gamma(E) = \sup\{|f'(\infty)| : f \in H(E)\}$.

In [6], it was conjectured that the convergence of a "generalized Melnikov's series" implies the unit ball of $R(X)$ admits ϕ as a modulus of approximate continuity at a point. We are unable to prove this. Using a well known localization procedure and Melnikov's estimate for Cauchy integrals [2], however, we can get a weaker result (Theorem 4). Hayashi [3] has obtained a similar result independently when he considered the case of the first order bounded point derivations.

Received by the editors on September 13, 1977, and in revised form on December 19, 1977.