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# SINGULAR PERTURBATIONS FOR DIFFERENCE EQUATIONS

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ABSTRACT. This paper discusses singular perturbations for second-order linear difference equations with a small parameter. It is found that there exhibits boundary layer behavior for the two-point boundary-value problem as well as for the final-value problem, but not for the initial-value problem. In contrast to problems for differential equations, a boundary layer exists only at the right end point. By virtue of a stretching transformation, a formal procedure is developed for treating such problems, and the justification of this procedure is established through a discrete maximum principle.

1. **Introduction.** We consider here the singular perturbation for the second-order linear difference equations. More precisely we consider the boundary value problem $(P_\epsilon)$ defined by

$$\epsilon Y_{k+2} + a_k Y_{k+1} + b_k Y_k = 0 \quad (k = 0, 1, 2, \cdots, N-2), \tag{1.1}$$

and

$$Y_0 = \alpha, \qquad Y_N = \beta. \tag{1.2}$$

Here $\epsilon > 0$ is a small parameter; $a_k$ and $b_k$ are non-zero discrete functions which are assumed to be bounded; $\alpha$ and $\beta$ are given constants. We shall study the asymptotic behavior of the solution when $\epsilon$ approaches zero. By analogy with the ordinary differential equations [2], the problem $(P_\epsilon)$ is said to be *singular* in the sense that the *degenerate problem* (or the *reduced problem*) $(P_0)$,

$$a_k Y_{k+1} + b_k Y_k = 0 \qquad (k = 0, 1, 2, \cdots, N-2) \tag{1.3}$$

together with (1.2) has no solution.

Our goal here is to develop a procedure for treating such problems. In particular, we are interested in the boundary layer behavior as well as the feasibility of applying the method of inner and outer expansions to singular perturbation problems for the difference equations. We will see that in contrast to the differential equation problem,

$$\begin{gathered} \epsilon y'' + ay' + by = 0 \qquad (x_0 < x < x_N) \\ y(x_0) = \alpha \quad \text{and} \quad y(x_N) = \beta \end{gathered} \tag{1.4}$$

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