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A NEW FUNCTIONAL EQUATION WITH SOME SOLUTIONS†

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1. **Introduction.** Functional equations arising naturally in applied mathematics are not too common (see for example Aczel's treatise [1] on the general topic), and here we introduce a system springing from elementary concepts in queueing theory. In a previous paper [2] the authors have developed a system of generalized discrete probability distributions by considering equations of the form $x = y\,g(x)$, where $g(x)$ is the generating function of a random variable defined on the non-negative integers, and x, as a function of y, in general generates a Lagrangian probability generating function.

It now turns out that there is a tie up with the behavior of the random variable which describes the length of a busy period in a single server queueing system where the 'arrivals' follow some defined probability distribution. From general considerations of this situation, we were led to the functional equation

$$(1)\quad H(x, y) = 1 + xy(1 - xy)^{-1}\{1 - \psi(\lambda - \lambda xy)\}\, H(\psi(\lambda - \lambda xy), y),$$

where x and y are defined over the real domain, λ is a real number, and H, ψ are real, continuous and infinitely differentiable functions. The coefficients of the successive powers of x and y represent the probabilities of different lengths of busy periods for queues initiated by 1, 2, 3, $\cdots$ customers when $\psi(\lambda - \lambda xy)$ is a Laplace transform associated with the inter-arrival and inter-service density functions. The queues initiated by one customer have been considered in several studies; however it is a matter of common experience at medical clinics, automobile service stations and factory supply offices etc., that the First Busy Period is usually initiated by a larger number of customers than one. Thus the problem becomes a little more complex.

Our equation (1) is a specal case of a general functional equation

$$(2)\qquad H(x, y) = 1 + \phi(xy)\, H(\psi(xy), y),$$

where $\phi(xy)$ and $\psi(xy)$ are real and continuous functions of xy having power series expansions and such that $\phi(0) = 0$. The observation that

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