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FORMALLY SELF-ADJOINT QUASI-DIFFERENTIAL OPERATORS*

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Introduction. In the study of ordinary differential operators the class of formally self-adjoint differential expressions plays an important role. Such expressions generate symmetric operators in the L^2 spaces, and hence the well developed theory of symmetric and particularly self-adjoint operators in Hilbert space (or more generally in Banach spaces) can be applied to study the spectrum of such operators, among other things.

The classical definition of formal self-adjointness — see the book by Coddington-Levinson [3, p. 84] — is as follows: consider the nth order differential expression:

$$Ly = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y , \tag{1}$$

where

$$p_i \in C^i \text{ for } i = 0, 1, \cdots, n , \tag{2}$$

and the adjoint operator L^+ defined by

$$L^+ y = (-1)^n (\bar{p}_n y)^{(n)} + (-1)^{n-1} (\bar{p}_{n-1} y)^{(n-1)} + \cdots + \bar{p}_0 y. \tag{3}$$

The expression L is said to be formally self-adjoint if $L = L^+$.

It is well known — see Neumark [6] or Dunford and Schwartz [5, p. 1290] — that every formally self-adjoint differential expression L whose coefficients satisfy (2) is of the form

$$\sum_{j=0}^{[n/2]} (-1)^j (a_j y^{(j)})^{(j)} + \sum_{j=0}^{[(n-1)/2]} i[(b_j y^{(j)})^{j-1} + (b_j y^{j+1})^{(j)}] , \tag{4}$$

where a_j, b_j are real.

In particular every formally self-adjoint differential expression L with real coefficients satisfying (2) is of even order $n = 2m$ and has the form

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