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HOMOTOPY-ALGEBRAIC STRUCTURES

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In topology, there are many objects of study that consist of a space together with an "operation" on it. One may think of a topological group structure, an H-space structure, a homotopy self-equivalence, etc. One wishes to classify such operations up to homotopy and to consider the possible relations such an operation may satisfy. In this paper we provide a general framework to study these questions in terms of the Postnikov system of the space in question. Our model is the well-known fact that a space is an H-space if and only if its Postnikov invariants are primitive, and we are inspired by the work of Stasheff, [7].

The spaces we shall consider will be connected CW-complexes with basepoint. Let X be such a space, with x_0 its basepoint. Denote the cartesian product of n copies of X by X^n and let $T_1{}^n(X)$ be the subspace of X^n consisting of all points at least one of whose coordinates is the basepoint.

DEFINITION 1. An *(n-ary) operation* on X consists of a pointed continuous function $\phi : X^n \to X$.

Let $\mathcal{P}X$ denote the (Moore) free path-space of X, i.e., the set of all pairs (λ, r) such that $r \geqq 0$ and $\lambda : [0, r] \to X$ is continuous. We have two projections of $\mathcal{P}X$ onto X, π_0 and π_∞, given by $\pi_0(\lambda, r) = \lambda(0)$ and $\pi_\infty(\lambda, r) = \lambda(r)$. The basepoint of $\mathcal{P}X$ is taken to be the pair $(\lambda_0, 0)$ such that $\lambda_0(0) = x_0$.

DEFINITION 2. If $\phi, \psi : X^n \to X$ are operations, a *relation* between ϕ and ψ is a homotopy $R : X^n \to \mathcal{P}X$ such that $\pi_0 \circ R = \phi$ and $\pi_\infty \circ R = \psi$.

REMARK. Since $T_1{}^n(X)$ is retractile [3] in X^n, if ϕ and ψ agree on $T_1{}^n(X)$, then R may be chosen to remain fixed on $T_1{}^n(X)$.

DEFINITION 3. Suppose that $\phi : X^n \to X$ and $\phi_1 : X_1{}^n \to X_1$ are operations. A map $f : X \to X_1$ is called a *(ϕ, ϕ_1)-map* provided that there exists a homotopy $H : X^n \to \mathcal{P}X_1$ such that $\pi_0 \circ H = \phi_1 \circ f^n$ and $\pi_\infty \circ H = f \circ \phi$.

Observe that $\mathcal{P}X$ is a functor in X, i.e., that given $f : X \to Y$ we may define $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ by $\mathcal{P}f(\lambda)[t] = f(\lambda(t))$.

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