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# APPROXIMATION INDUCED BY A FOURIER SERIES

CHARLES K. CHUI

1. **Introduction.** Let $X$ be a normed vector space and $S$ be a subset of $X$ such that the vector space generated by $S$ is dense in $X$. Suppose that $f$ is a mapping from $S$ into $X$ and $V(f; S)$ is the vector space generated by the set $\{f(x) : x \in S\}$. We ask the following question: For what $S$ and $f$ is $V(f; S)$ dense in $X$? For instance, for the Banach space $C[0, 1]$ of continuous functions on the closed unit interval $[0, 1]$ with the supremum norm, we can take $S = \{1, t, t^2, \cdots\}$. If the mapping $f$ from $S$ into $C[0, 1]$ is given by $f(t^k) = t^{\alpha k}$, $k = 0, 1, \cdots$, where $\alpha > 0$, then by the Stone-Weierstrass theorem we see that $V(f; S)$ is dense in $C[0, 1]$. Other sets $S$ and other mappings $f$ for the Banach space $C[0, 1]$ have been considered by Korevaar [3] and Luxemburg [4]. In this note, we consider the Banach spaces $L^p = L^p(T)$ and $C = C(T)$, where $T$ will always denote the unit circle $|z| = 1$ in the complex plane, and we take $S = \{1, e^{it}, e^{-it}, e^{i2t}, e^{-i2t}, \cdots\}$. Our $f$ will be defined by some absolutely convergent Fourier series, and in particular, some exterior conformal maps. It should be mentioned that some related but a little different problems on Hilbert spaces have been considered by Hilding [1, 2] and Pollard [5].

2. **Approximation Induced by a Fourier Series.** Let $\mathcal{G}$ denote the class of all Fourier series $\sum_{-\infty}^{\infty} a_k e^{ikt}$ with

$$\sum_{k=2}^{\infty} (|a_k| + |a_{-k}|) \leqq \big|\, |a_1| - |a_{-1}| \,\big|. \tag{1}$$

Any Fourier series of class $\mathcal{G}$ converges uniformly to a continuous function on $T$, and we also say that this limit function belongs to class $\mathcal{G}$. Hence,

$$f(e^{it}) = \sum_{k=-\infty}^{\infty} a_k e^{ikt} \tag{2}$$

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