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A NOTE ON THE USE OF A CONTINUED FRACTION FOR PERTURBATION THEORY

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ABSTRACT. An implicit eigenvalue-eigenvector problem usually solved using perturbation theory is shown to lead to a nonperturbation solution in terms of a continued fraction.

In this note we summarize a method to obtain a nonperturbative solution to the implicit eigenvalue-eigenvector problem arising in nonrelativistic quantum mechanics [1]:

$$[E - e - t(E)P_0]\ |\alpha\rangle = 0. \tag{1}$$

In equation (1), $t(E)$ is a linear operator in the Hilbert space spanned by the discrete eigenfunctions of the operator H_0 and $t(E)$ satisfies

$$t(E) = V + VT_0(E)t(E). \tag{2}$$

In equation (2), $T_0(E) = Q_0(E - H_0)^{-1}$ and $P_0 + Q_0 = 1$, $P_0{}^2 = P_0$, $Q_0{}^2 = Q_0$ and $P_0|\alpha\rangle = |\alpha\rangle$, $Q_0|\alpha\rangle = 0$. Now if we solve equation (2) for $t(E)$ we obtain

$$t(E) = V^{1/2}(1 - V^{1/2}Q_0(E - H_0)^{-1}V^{1/2})^{-1}V^{1/2}. \tag{3}$$

By inspection of the expression for t in equation (3) we see that tP_0 can be expressed as the composition of two successive linear fractional transformations of operator argument:

$$\begin{aligned} \mathcal{T}_0(U) &= V^{1/2}(1 - U)^{-1}V^{1/2}P_0 \\ \mathcal{T}_1(U) &= V^{1/2}Q_0(U - H_0)^{-1}V^{1/2}. \end{aligned} \tag{4}$$

On the other hand by using the Cauchy formula and multiplying on the left by $\langle\alpha|$ equation (1) may be transformed into

$$\langle\alpha|E - e - \frac{1}{2\pi i}\oint \frac{t(\epsilon)P_0\, d\epsilon}{\epsilon - E}|\alpha\rangle = 0. \tag{5}$$

The contour includes the eigenvalue E. If we then introduce an equivalent operator h such that $E|\alpha\rangle = (e + h)|\alpha\rangle$, then equation (5) may be written as

$$\langle\alpha|E - e - \frac{1}{2\pi i}\oint \frac{t(\epsilon)P_0 d\epsilon}{\epsilon - e - h}|\alpha\rangle = 0. \tag{6}$$

This equation will be satisfied if