ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 2, Spring 1974

CONTINUED FRACTIONS IN BANACH SPACES

T. L. HAYDEN

We assume that the proper generalization of continued fractions to complex Banach spaces is the limit of holomorphic maps generated by composition of linear fractional maps. The deepest results are those of Mac Nerney [10], [11], in which a major part of the work of H. S. Wall on positive definite continued fractions is generalized to $B(H)$, the space of bounded operators on a Hilbert Space. In case the map is from the plane into $B(H)$, Mac Nerney also obtains connections with moment problems. Russian work on linear fraction maps and applications may be found in [9].

In $B(H)$ we are forced to use symmetry in our definition due to the lack of commutativity. Mac Nerney observed this in his work and we note that the Möbius transformations in $B(H)$ have the symmetric form $L \cdot (I - BB^*)^{-1/2}(Z + B)(I + B^*Z)^{-1}(I - B^*B)^{1/2}$ where $*$ denotes the adjoint, L is a linear isometry, and Z and B are in the open unit ball [7]. This lack of commutativity makes the recursion relations for continued fractions extremely complicated [10, p. 675], we shall try to establish convergence by other means.

We consider the continued fraction $A_1/I + A_2/I + \cdots$ to be generated by transformations $t_n(w) = A_n(I + w)^{-1}$ and define the continued fraction as $\lim_{n\to\infty} T_n(0) = \lim_{n\to\infty} t_1 \cdot t_2 \cdots t_n(0)$. We prefer the limit in the uniform operator topology on $B(H)$, but in some cases only strong limits are obtained.

Suppose first that each A_i is a positive operator, $(A_ix, x) \geqq 0$, for each x in H. Since the product of positive operators may not be positive unless they commute, and since we wish the limit to be a positive operator we modify the t_n to $t_n(w) = A_n(I + w)^{-1} A_n^*$ or $(\sqrt{A_n})(I + w)^{-1} (\sqrt{A_n})$. Then if $w \geqq 0$, $t_n(w) \geqq 0$. The usual order for operators $A \leqq B$ means $(Ax, x) \leqq (Bx, x)$.

PROPOSITION. *If $A_i \geqq 0$, then the odd approximants decrease and exceed the even approximants which increase. Hence the even and odd approximants converge strongly. If the A_i are uniformly bounded and commute the continued fraction converges in the uniform operator topology to a positive operator.*

If the A_i commute, then via the Gelfand map the algebra of operators generated by A_i is isomorphic to continuous real valued functions and the positive operators correspond to positive functions. Since the