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MODIFIED MOMENTS AND GAUSSIAN QUADRATURES*

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A number of the speakers at this conference have referred to or made use of the close connection between Padé approximants, continued fractions, moment theory, orthogonal polynomials, and (Gaussian) quadrature formulas. I want to discuss an application of a recent development in the theory (and practice) of Gaussian quadratures which may well have repercussions in other of these areas, namely, the *modified moments procedure* discussed recently by Sack and Donovan [14] and Gautschi [1].

1. **The Quadrature Problem.** In the language of Gaussian quadratures, the central problem I want to consider is the following: Given a finite number $(N + 1)$ of *power* moments, $\mu_0, \mu_1, \cdots, \mu_N$ of an *unknown* probability density $G(x)$:

$$\mu_k \equiv \langle x^k \rangle \equiv \int_a^b x^k G(x)\, dx \quad (-\infty \leqq a < b \leqq +\infty), \tag{1.1}$$

estimate or bound the average of a *known* function $F_\tau(x)$:

$$\langle F_\tau(x) \rangle \equiv \int_a^b F_\tau(x)\, G(x)\, dx, \tag{1.2}$$

where $F_\tau(x)$ may depend on a parameter τ as well as upon x. This is accomplished by replacing the integral by a quadrature formula:

$$\langle F_\tau(x) \rangle = \sum_1^n w_i F_\tau(x_i) + \Delta \tag{1.3}$$

where the abscissas x_i and weights w_i are determined by the condition that they give the moments correctly:

$$\sum_{i=1}^n w_i x_i^k = \mu_k \ (k = 0, \cdots, N). \tag{1.4}$$

When both a and b in (1.1) and (1.2) are finite, four distinct quadratures are useful: with no preassigned abscissas (Gauss), with one abscissa fixed at either a or b (Radau), or with abscissas fixed at both a and b (Lobatto). The number of abscissas and weights is chosen

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