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THE ROLE OF THE POLE IN RATIONAL APPROXIMATION

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The purpose of this paper is to present some old and some recent results that seem to indicate the current direction of growth of the theory of approximation by rational functions. An older result is in Walsh [**10**, § 8.7].

THEOREM 1. *In the z-plane, let the Jordan curve C_0 contain in its interior the Jordan curve C_1, and let the two curves bound the region B. Let the function $U(z)$ be harmonic in B, continuous on C_0 and C_1, equal to zero and unity on those respective curves. Let C_r denote generically the level locus $U(z) = r$, $0 < r < 1$, in B. Then there exist points α_{nk} $(k = 1, 2, \cdots, n)$, and points β_{nk} $(k = 1, 2, \cdots, n + 1)$ equally spaced on C_0 and C_1 respectively with regard to the conjugate of $U(z)$ in C, so that if $f(z)$ is a function analytic throughout the closed interior of C_μ, and if $r_n(z)$ denotes the rational function of degree n with poles in the α_{nk} and interpolating to $f(z)$ in the β_{nk}, then we have* $(\mu' > \mu)$

$$\limsup_{n\to\infty}\,[\max|f(z) - r_n(z)|, z \text{ on and within } C_{\mu'}]^{1/n} \leqq e^{-2\pi(\mu'-\mu)/\tau}, \tag{1}$$

where $\tau = \int_{C_r} (\partial U/\partial \nu)\, ds$, *$\nu$ being the interior normal on C_r.*

For approximation by polynomials in a Jordan arc we have [**11**, § 2.3].

THEOREM 2. *Let C be an analytic Jordan arc in the z-plane, let $f(z)$ be defined on C, and let $z = \phi(w)$ map C one-to-one and conformally onto the line segment S: $-1 \leqq w \leqq 1$. Then a necessary and sufficient condition that there exist polynomials $P_n(z)$ of respective degrees n satisfying*

$$|f(z) - p_n(z)| \leqq A/n^{p+\alpha}, z \text{ on } C, 0 < \alpha < 1, \tag{2}$$

is that $f[\phi(\cos\theta)]$ possess on S a pth *derivative with respect to θ which satisfies a Lipschitz condition there with respect to θ.*

If C itself is S, Theorem 2 becomes the classical results in the real

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