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RATIONAL APPROXIMATIONS WITH APPLICATIONS TO THE SOLUTION OF FUNCTIONAL EQUATIONS*

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1. **Introduction.** The subject of Padé approximations was begun about a century ago. Although considerable progress has been made in recent times on various aspects of the subject, analytical expressions for the pertinent polynomials of the Padé matrix table, recurrence formulas for the polynomials and general pointwise convergence theorems are known for only a small class of functions.

In this exposition, we take the approach of developing rational approximations for a large class of functions which can be defined by series expansions, differential equations, integral transforms, etc., without insisting on the definition which characterizes the Padé approximation. In this fashion we show how closed form analytical expressions for the polynomials emerge naturally. Also, a general theorem concerning the existence of recurrence relations is proved. Further, by seizing upon the functional properties noted above, the error in the approximation process can be characterized in an analytic manner, from which important data on convergence and assessment of the error can be deduced. In some instances, these rational approximations do reduce to familiar Padé approximations for a certain class of well known functions.

2. **Rational Approximations for Functions Defined by a Series.** Suppose

$$F(z) = \sum_{r=0}^{n-1} b_r z^r + R_n(z), \tag{1}$$

where b_r is independent of n and z and $R_n(z)$ is the error in the above polynomial approximation to $F(z)$. Expressions of the type (1) can be obtained in a variety of ways — viz. solutions of differential equations and from integral transforms such as those of Laplace and Mellin-Barnes integrals. In (1) replace n by $k + 1 - a$, $a = 0$ or $a = 1$, multiply both sides by $A_{n,k}\gamma^{-k}$ and sum from $k = 0$ to $k = n$. Here γ is a free parameter, $A_{n,k}$ is independent of z and γ, and $A_{n,k} = 0$ if $k > n$. Then

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