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CONSTRUCTION OF THE REALS VIA ULTRAPOWERS

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1. **Introduction.** In courses on real analysis it is convenient and not unusual to define the real number system R axiomatically as a "complete ordered field", that is, as an ordered field which satisfies the least upper bound axiom. Then all other properties of the reals (e.g., uniqueness up to isomorphism, the Heine-Borel theorem, etc.) can be proved directly from these axioms and it is not necessary to argue from some particular "construction" such as Dedekind cuts or equivalence classes of Cauchy sequences of rationals. (This latter method of construction we shall refer to as "Cantor's method".) In particular, it is easily proved that an ordered field satisfies the least upper bound axiom iff it is archimedean ordered and Cauchy sequentially complete.

Of course this axiomatic definition of the reals leaves open the question of the *existence* of a complete ordered field; and for this purpose a construction of some sort can hardly be avoided. Furthermore, Cantor's method can hardly be improved upon for its simplicity, directness and freedom from transfinite existence principles (such as the axiom of choice or Zorn's lemma). Cantor's method will certainly remain an important method of "completion".

Nevertheless, we sketch here another method for constructing the reals from the rationals Q which uses the notion of an "ultrapower" (introduced in [8] and developed in [3]; but see also [5]) and which we feel is not without interest even though it is neither shorter nor simpler than Cantor's method. It has the additional disadvantage that it depends on the following transfinite existence principle, which we shall refer to as "the ultrafilter hypothesis".

(U) *Every filter is contained in some ultrafilter.*

On the positive side we make three points which we consider to be pedagogical advantages of our construction.

1. Transfinite principles of some sort seem to be here to stay, and

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