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A CONDITION ALLOWING THE REDUCTION OF THE GENUS OF A HEEGAARD SPLITTING

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In [1] it is shown that every 3-manifold can be given a combinatorial triangulation. It follows from this that any orientable, closed 3-manifold M can be represented as $H \cup H'$ where H and H' are solid tori of genus n (i.e., homeomorphic to regular neighborhoods of compact connected graphs with Euler characteristic $1 - n$) and $H \cap H' = \partial H = \partial H' = T$ is an orientable surface of genus n. This is called a Heegaard splitting of genus n for M. It is known that the 3-sphere S^3 is the only simply connected such manifold with a Heegaard splitting of genus 1. (Manifolds with Heegaard splittings of genus 1 are called lens spaces.) Thus, a possible approach to the Poincaré conjecture is to find conditions under which the genus of a Heegaard splitting for any homotopy 3-sphere might be reduced. We give here (Theorem 2) one such set of conditions.

All spaces considered will be polyhedra and all maps will be piecewise linear. The following characterization is an easy consequence of Dehn's Lemma [3], the loop theorem [4] and Poincaré duality.

PROPOSITION 1. *Let H be a compact, connected 3-manifold with connected boundary. Assume $\pi_1(H)$ is a free group of rank n. Then H is a solid torus of genus n if and only if H can be embedded in R^3.*

Now consider a compact, orientable surface F with nonempty boundary. Let $r = g(F)$ be the genus of F and s be the number of boundary components. Then F has Euler characteristic $\chi(F) = 2 - 2r - s$ and $\pi_1(F)$ is free of rank $2r + s - 1$. Hence, $F \times [0, 1]$ is a solid torus of genus $2r + s - 1$.

THEOREM 2. *Suppose $H \cup H'$ is a Heegaard splitting of genus n for the closed 3-dimensional manifold M. Assume that F is a compact, connected, orientable surface with nonempty boundary and that $h : F \times [0, 1] \to H'$ is a homeomorphism such that $h(\partial F \times 0)$ bounds an orientable, not necessarily connected, surface G properly embedded in H with $\chi(G) > 1 - n$. Then there is a Heegaard splitting for M of genus less than n.*

PROOF. Assume that G is chosen so that $\chi(G)$ is maximal. (Note that

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