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ON $\Delta - m$ SETS, ALMOST PERIODIC FUNCTIONS AND GROUP TOPOLOGIES

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1. **Introduction.** Markov [1], in a paper in 1933, introduced a combinatorial property of subsets of the real line which he used to prove that certain stability properties of solutions of differential equations implied their almost periodicity. The only property of these sets that he used is the one that we reproduce below as Theorem 1. It turns out that this combinatorial property makes sense and is useful in arbitrary groups.

2. **$\Delta - m$ sets.**

DEFINITION 1. A set S in a group $(G, +)$ is called a $\Delta - m$ *set* (m a positive integer) in case (1) given elements of G, $t_1, t_2, \cdots, t_{m+1}$, not necessarily distinct, there exist $i \neq j$ such that $t_i - t_j \in S$; and (2) S is symmetric with respect to the identity of G, i.e., $-S = S$.

We can make several remarks. Since all the t_i's may be equal, $0 \in S$ for all $\Delta - m$ sets. Furthermore, the choice $t_1 = t$, $t_2 = 0$ implies that every $\Delta - 1$ set contains t or $-t$ for t an arbitrary element of G. Since S is symmetric, it contains both. Thus the only $\Delta - 1$ set is G itself.

We first restrict ourselves to the additive group of reals. It may be verified that $E_k = \bigcup_{n=-\infty}^{\infty} [n - 1/k, n + 1/k]$ $(k \geqq 3)$ is a $\Delta - (k - 1)$ set but not a $\Delta - (k - 2)$ set.

As these examples indicate, $\Delta - m$ sets in $\mathbf{R}$ can contain gaps; however, in some sense they cannot be too sparse. The next two results make this precise.

THEOREM 1. *Every $\Delta - m$ set in $\mathbf{R}$ is relatively dense, i.e., if S is a $\Delta - m$ set, then there exists an $L(S) > 0$ such that $S \cap [t, t + L(S)] \neq 0$ for all real t.*

PROOF. The proof is by induction on m. The statement for $\Delta - 1$ sets is clear since $\mathbf{R}$ is the only such set. Assume the theorem true for $\Delta - (m - 1)$ sets. Let S be a $\Delta - m$ set. If S is also a $\Delta - (m - 1)$

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