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ALGEBRAS OF INTEGRABLE FUNCTIONS. II

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1. **Introduction.** Morera's theorem in complex function theory raises the possibility that this theory can be based on integration rather than differentiation. Heffter [1], Macintyre and Wilbur [9] and this author [7] have given such a development. In this paper a theory of integrable functions will be developed in a more general context of operator valued functions wherein the functions no longer need be analytic.

Let K denote the complex plane. For $a \in K$, set $A_a(z) = az$ for all $z \in K$. Then A_a is a bounded linear transformation of K, thought of as a real Euclidean space E_2 into itself. Set $T_2{}' = \{A_a;\, a \in K\}$. Let f be a continuous function on an open set S in E_2 into the space B_2 of bounded linear transformations of E_2 into itself, and let P be a path (rectifiable arc) in S with endpoints α and β. Then for any subdivision $\alpha = z_1 < \cdots < z_{n+1} = \beta$ of P, a Riemann sum, the vector $R = \sum_{i=1}^{n} f_{z_i}(z_{i+1} - z_i)$ can be formed. If range f lies in $T_2{}'$, then, for $z \in \mathrm{S}$, $f_z = f(z) = A_{\phi(z)}$ for some $\phi(z) \in K$, and we may write $R = \sum_{i=1}^{n} \phi(z_i)(z_{i+1} - z_i)$. Taking the limit as the norm of the subdivision defining R approaches zero, we obtain the vector $\theta = \int_{\alpha_P}^{\beta} f(z)\, dz = \int_{\alpha_P}^{\beta} f_z(dz)$. If range $f \subseteq T_2{}'$, we can interpret θ as the complex number $\int_{\alpha_P}^{\beta} \phi(z)\, dz$.

f is said to be integrable if for all closed paths (rectifiable simple closed curves) $C \subseteq \mathrm{S}$, we have $\int_C f(z) dz = \int_C f_z(dz) = 0$. If range $f \subseteq T_2{}'$, then $\int_C \phi(z)\, dz = 0$ for all closed paths $C \subseteq \mathrm{S}$, and by Morera's theorem ϕ is analytic; consequently, f is itself Fréchet differentiable, where $f_z{}'$ is a linear transformation of E_2 into B_2 for $z \in \mathrm{S}$.

The general case studied in this paper is obtained by replacing E_2 by an arbitrary real Euclidean space E of dimension p, $p > 1$. Let T be a *commutative* subalgebra of the Banach algebra B of bounded linear transformations of E into E and let F be the family of continuous integrable functions on open subsets of E into T.

Let E' be a finite-dimensional commutative Banach algebra with identity over the reals and for $a \in E'$, set $A_a(t) = at$ for $t \in E'$. Set $T' = \{A_a;\, a \in E'\}$.

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