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SPECTRAL REPRESENTATION OF SELFADJOINT EXTENSIONS OF A SYMMETRIC OPERATOR

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ABSTRACT. It is shown that the spectral multiplicity of a minimal selfadjoint extension A of a simple closed symmetric operator A_1 with deficiency indices m, n cannot exceed $m + n$. In the case that A_1 has deficiency indices 1, 1, it is shown that any minimal selfadjoint extension A can be represented as a multiplication operator in a space $L_P^2(-\infty, \infty)$, where $\mathrm{P}(t)$ is a 2 by 2 nondecreasing Hermitian matrix function of t. In this case the spectrum and spectral multiplicity of A are studied by use of $\mathrm{P}(t)$ and its relation to the matrix $\Phi(\lambda) = \int_{-\infty}^{\infty}[(t-\lambda)^{-1} - t(1+t^2)^{-1}]\,d\mathrm{P}(t)$, where λ is a complex variable. A criterion is given for when the spectral multiplicity of A is two and for when it is one. It follows from this criterion that if A_1 has a selfadjoint extension A_0 in the original space with a singular spectral function, then the spectral multiplicity of any minimal selfadjoint extension A is one.

1. **Introduction.** Let A_1 be a closed symmetric operator with deficiency indices m, n in a Hilbert space $\mathfrak{H}_1$. We suppose that A_1 is *simple*, i.e., that A_1 does not have a reducing subspace in which it is selfadjoint. A selfadjoint operator A in a Hilbert space $\mathfrak{H}$ is called an *extension* of A_1 if $\mathfrak{H}_1 \subseteq \mathfrak{H}$ and $A_1 \subset A$. A selfadjoint extension A is said to be *minimal* if the only subspace of $\mathfrak{H} \ominus \mathfrak{H}_1$ which reduces A is $\{0\}$. In this article it is shown that the spectral multiplicity of a minimal selfadjoint extension of A_1 cannot exceed $m + n$. In the case that A_1 has deficiency indices 1, 1 a spectral representation is given for any minimal selfadjoint extension A of A_1 in the form of a multiplication

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