ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 1, Winter 1972

REMARKS ON HOMOLOGICAL DIMENSIONS

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1. **Introduction.** It has been noticed by the author and others that there are many ways in which the homological dimension of a module can be defined. The author has noticed this particularly with regard to algebraic geometry and specifically with regard to the following three theorems:

a. *A local ring is regular if and only if it is a local ring of finite global dimension.*

b. *The quotient ring of a regular local ring at any prime ideal is also a regular local ring.*

c. *Regular local rings are unique factorization domains.*

It is, therefore, the purpose of this article to present four of the more well-known definitions of homological dimension and to show that these four are equivalent if one restricts oneself to the class of finitely generated modules over Noetherian local rings.

It will be assumed throughout this article that R is a commutative ring with unit. It will be clear, however, that one could assume that R is not necessarily commutative if one would consistently use left R-modules or right R-modules throughout.

2. The first definition of homological dimension that will be presented is due to Cartan and Eilenberg [1].

DEFINITION. An R-module A has homological dimension n (n an integer $\geqq 0$) if and only if

(i) there is a projective resolution of A of the form $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$, and

(ii) there does not exist such a projective resolution for A with fewer terms.

In this case we write $dh_R(A) = n$. If no such projective resolution exists for A with a finite number of terms we write $dh_R(A) = \infty$.

Another definition of homological dimension is due to Kaplansky [2]. Here it is said first that two R-modules A and B are *equivalent* ($A \sim B$) if and only if there exist projective R-modules P and P' such that $A \oplus P \cong B \oplus P'$. This is easily seen to be an equivalence relation if one observes that the direct sum of projective R-modules is again a projective R-module.

Received by the editors March 16, 1970.
AMS 1970 *subject classifications.* Primary 14A05, 13D05.