ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 1, Winter 1972

GENERATING SETS FOR A FIELD AS A RING EXTENSION OF A SUBFIELD

ROBERT GILMER[1]

1. **Introduction.** Suppose that F is a subfield of the field L. L can be considered as a field extension of F, as a ring extension of F, or as a vector space over F, and hence the term *generating set for L over F* may mean either (1) a subset S of L such that $L = F(S)$, or (2) a subset S of L such that $L = F[S]$, or (3) a subset S of L such that S spans L as a vector space over F. Of course, a generating set in the sense of (3) is a generating set in the sense of (2), and if (2) holds for S, then (1) holds for S. Moreover, (1) and (2) are equivalent if L/F is algebraic.

In this paper we are primarily concerned with *ring generating sets* for L/F—that is, subsets of L satisfying (2). We denote by $\rho(L, F)$ the smallest cardinal number α such that there is a ring generating set for L over F of cardinality α. A theorem of Becker and Mac Lane [1] implies that if $[L : F]$ (the cardinality of a vector space basis for L over F) is finite, and if L/F is inseparable, then $\rho(L, F) = r$, where $[L : L^p(L_s)] = [L : L^p(F)] = p^r$ and L_s is the set of elements of L which are separable over F. We prove (Theorem 4) that $\rho(L, F) = [L : F]$ if L/F is algebraic but not finite, and $\rho(L, F) = |L|$ if L/F is not algebraic. In particular, $\rho(K, F) \leqq \rho(L, F)$ if K is a subfield of L containing F.

If $L = F[S]$ and if K is a subfield of L containing F, we prove (Corollary 3) that $K = F[T]$ where $|T| \leqq |S|$, and except in the case when L/F is finite algebraic and K/F is not purely inseparable, it is true that if $K = F[S_0]$, then there is a subset T of S_0 such that $K = F[T]$ and $|T| \leqq |S|$. In §4 we conclude with some observations concerning $\rho(L, F)$ and $[L : F]$.

2. **Preliminaries on cardinality.** We begin by listing some results on cardinal numbers which we shall need in the sequel.

RESULT 1. *If N is a regular multiplicative system in the infinite commutative ring R, then $|R_N| = |R|$; in particular, $|R| = |T|$, where T is the total quotient ring of R.*

Received by the editors August 20, 1970.

AMS 1970 *subject classifications*. Primary 12E99, 12F99; Secondary 13A99.

[1]This research was supported by National Science Foundation Grant GP-19406.