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A CLOSURE PROPERTY OF REGRESSIVE ISOLS

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0. **Introduction.** Let ϵ^*, ϵ, Λ, Λ_R and Λ^* denote the collections of all integers, nonnegative integers, isols, regressive isols, and isolic integers respectively. Let $f(x_1, \cdots, x_n)$ be a recursive function, and let f_Λ denote the canonical extension of f to a mapping from Λ^n into Λ^*. Let Δ be any subcollection of Λ. We say that Δ *is closed under* f if $f_\Lambda(\Delta^n) \subseteq \Delta$. A. Nerode proved in [12] that Λ is closed under f if and only if f is almost recursive combinatorial. In [2], J. Barback showed that if f is a recursive function of one variable, Λ_R is closed under f if and only if f is eventually increasing. The purpose of this paper is to characterize the class of recursive functions of two variables mapping $\Lambda_R{}^2$ into Λ_R. The class obtained is surprisingly limited; it consists primarily of functions of the form min $(f(x), g(y))$ where min (x, y) is the usual minimum function and $f(x)$ and $g(y)$ are eventually increasing and recursive. A precise statement of the main result requires the following two definitions. $f(x, y)$ will be called *flat* if there is a (recursive) function $g(x, y)$ such that $g(x, y) = 0$ for all but finitely many pairs $(x, y) \in \epsilon^2$ and $f(x, y) = \sum_{i=0}^{x} \sum_{j=0}^{y} g(i, j)$ for all $(x, y) \in \epsilon^2$. $f(x, y)$ will be called *reducible to the case of a single variable* if (i) there exist eventually increasing recursive functions $f_i(y)$, $i = 0, \cdots, m$, such that $f(x, y) = f_x(y)$ for $x \leqq m$ and $f(x, y) = f_m(y)$ for $x > m$, or (ii) condition (i) holds with the roles of x and y interchanged. The main result is the following:

Λ_R is closed under a recursive function $f(x, y)$ if and only if there is an $n \in \epsilon$ such that:

(1) For $i \leqq n$, $f(i, y)$ is an eventually increasing function of y and $f(x, i)$ is an eventually increasing function of x,

(2) $f(x + n, y + n) = m(x, y) + c_1(x, y) - c_2(x, y)$ for $x, y \in \epsilon$, where c_1 and c_2 are flat recursive functions and $m(x, y)$ is either (i) reducible to the case of a single variable or (ii) of the form min $(g(x), h(y))$, where $g(x)$ and $h(y)$ are eventually increasing recursive functions of one variable.

Functions mapping $\Lambda_R{}^2$ into Λ_R have a natural use as Skolem func-

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