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LARGE ABELIAN SUBGROUPS OF SOME INFINITE GROUPS

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ABSTRACT. A generalization of the following conjecture of W. R. Scott is proved. If $(H_\alpha)_{\alpha \leqq \delta}$ is a well-ordered descending chain of subgroups of a group with the property that $H_\beta = \bigcap_{\alpha<\beta} H_\alpha$ for limit ordinals, then $[H_0 : H_\delta] \leqq \prod_{\alpha<\delta} [H_\alpha : H_{\alpha+1}]$. Using this, we show that the members of certain classes of infinite groups are guaranteed to have large abelian subgroups.

1. **Introduction.** Following Kurosh [5, p. 171], a totally ordered system $\mathcal{U}$ of subgroups of a group is said to be complete if for an arbitrary subsystem of $\mathcal{U}$, the unions and the intersections of the subgroups forming the subsystem belong to $\mathcal{U}$. W. R. Scott [8, p. 21] has conjectured that, if $(H_\alpha)_{\alpha \leqq \delta}$ is a well-ordered descending complete system of subgroups of a group H_0, then

$$[H_0 : H_\delta] \leqq \prod_{\alpha<\delta} [H_\alpha : H_{\alpha+1}]. \tag{1}$$

In a private communication, Scott has shown that this is indeed true for $\delta = \omega$, the first infinite ordinal, and has stated that under these same conditions both he and, independently, A. Kruse have proved that

$$[H_0 : H_\delta] \leqq \left[\prod_{\alpha<\delta} [H_\alpha : H_{\alpha+1}] \right]^{|\delta|}.$$

In §3 we shall establish a generalized form of Scott's conjecture from which the latter can be deduced. In addition, we shall find a lower bound for $[H_0 : H_\delta]$ which will be useful in §4.

If m is a cardinal number, we define $\exp m = \exp^1 m = 2^m$. Inductively, if n is any positive integer, we define $\exp^{n+1} m = \exp \exp^n m$. In §4 we utilize equation (1) to investigate the existence of large abelian subgroups of certain infinite groups. For example, C. R. Kulatilaka has shown [4, p. 241] that every infinite SI^*-group G

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